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STABLE MATCHING IN LARGE ECONOMIES

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We study stability of two-sided many-to-one matching in which firms' preferences for workers may exhibit complementarities. Although such preferences are known to jeopardize stability in a finite market, we show that a stable matching exists in a large market with a continuum of workers, provided that each firm's choice is convex and changes continuously as the set of available workers changes. We also study the existence and structure of stable matchings under preferences exhibiting substitutability and indifferences in a large market. Building on these results, we show that an approximately stable matching exists in large finite economies. We extend our framework to ensure a stable matching with desirable incentive and fairness properties in the presence of indifferences in firms' preferences.

KEYWORDS: Two-sided matching, stability, complementarity, strategy-proofness, large economy.

1. INTRODUCTION

SINCE THE CELEBRATED WORK by Gale and Shapley (1962), matching theory has emerged as a central tool for analyzing the design of matching markets. A key concept of the theory is "stability"—the requirement that there be no incentives for participants to "block" (i.e., side-contract around) a prescribed matching. Eliminating blocks keeps markets robust and promotes their long-term sustainability (Roth (2002)). Even when strategic blocking is not a concern, as in the case of public school matching, stability is desirable from the

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fairness standpoint, for it eliminates so-called justified envy (Balinski and Sönmez (1999), Abdulkadiroğlu and Sönmez (2003)).[1]

Unfortunately, a stable matching exists only under restrictive conditions. It is well known that in two-sided many-to-one matching, stability is not guaranteed unless the preferences of participants—for example, firms—are *substitutable*.[2] In particular, the presence of complementary preferences can lead to instability. This is a serious problem, given the pervasiveness of complementary preferences. It is not uncommon for firms to seek workers with complementary skills. In professional sports leagues, teams demand athletes that complement one another in terms of their skills and roles, etc. Some public schools in New York City seek diversity in their student bodies in terms of students' skill levels.[3] U.S. colleges tend to assemble classes that are complementary and diverse in terms of their aptitudes, life backgrounds, and demographics. To better organize such markets, one must understand the extent to which stability can be achieved in the presence of such complementarities; otherwise, the applicability of matching theory will remain severely limited.[4]

This paper takes a step forward in accommodating preference complementarities and other forms of general preferences. In light of the existing non-existence results, this requires us to weaken the notion of stability in some way. Our approach is to consider a large market. Specifically, we consider a market that consists of a continuum of workers on one side and a finite number of firms with a continuum of capacities on the other. We then ask whether stability can be achieved in an "asymptotic" sense—that is, whether participants' incentives for blocking disappear as the economy grows large and approaches the continuum economy in the limit. This weakening preserves the original spirit of stability: as long as the incentive for blocking is sufficiently weak, the instability and fairness concerns will not be a serious obstacle to organizing matching markets.

Large market models are also of interest since many real-world matching markets are large. School choice in a typical urban setting involves tens of thousands of students. Medical matching involves approximately 35,000 and 9,000 doctors in the United States and Japan, respectively. Aside from addressing complementary preferences, a large market model also allows us to address several outstanding issues in finite markets. One such issue is the multiplicity of stable matchings. While the set of stable matchings can be large in finite economies, there is a sense in which the set shrinks as the market grows large. Indeed, Azevedo and Leshno (2016) established that a stable matching is generically unique in a continuum economy when firms have so-called *responsive* preferences, a special case of substitutable preferences. To what extent such a result applies to more general preferences is an interesting issue that can be explored in a large market setting.

[1]In the school choice context, a student justifiably envies another if the former prefers the latter's school assignment to his and has a higher priority at that school. If a student has a sense of entitlement for his priority (e.g., when it is given by his test score or grade), eliminating justified envy appears to be important.

[2]Substitutability here means that a firm's demand for a worker never grows when more workers are available. More precisely, if a firm does not wish to hire a worker from a set of workers, then it never wishes to hire that worker from a larger (in the sense of set inclusion) set of workers.

[3] The so-called Educational Option programs in New York City high schools seek to fill 16% of seats with high reading performers (as measured by the score on the 7th grade standardized reading test), 68% of seats with middle reading performers, and the remaining 16% of seats with low reading performers (see Abdulkadiroğlu, Pathak, and Roth (2005)).

[4]In particular, this limitation is important for many decentralized markets that might otherwise benefit from centralization, such as the markets for college and graduate admissions. Decentralized college admissions may entail inefficiencies and a lack of fairness (see Che and Koh (2016)). However, to centralize college admissions, one must know how to deal with the potential instability arising from colleges' complementary preferences.

In addition to accommodating complementary preferences, we allow firms to be indifferent over different groups of workers. Indifferences may arise from firms' limited observations about workers' characteristics or their unwillingness to distinguish workers based on certain characteristics. Indifferences are particularly common in school choice, as schools apply coarse priorities to ration their seats,[5] in which case school preferences encoding the priorities will exhibit indifference over students.

Our first result is to characterize a stable matching as a fixed point of a suitably defined correspondence over measures of workers available to firms. This correspondence is reminiscent of the tâtonnement process, in that it iteratively maps each profile of worker types (in measure) available to firms to a new profile of available workers after processing firms' optimal choices on the former profile. Using this characterization, we establish the existence of a stable matching in general environments. First, we show that a stable matching exists if firms' preferences exhibit continuity or, more precisely, if each firm's choice correspondence is upper hemicontinuous and convex-valued. This result is quite general because these conditions are satisfied by a rich class of preferences, including those exhibiting complementarities.[6] The existence is established by means of the Kakutani–Fan–Glicksberg fixed-point theorem—a generalization of Kakutani's fixed-point theorem to functional spaces—which, to the best of our knowledge, is new to the matching literature.

Second, we obtain existence under the assumption of substitutable (but not necessarily continuous) preferences for firms. Substitutability means that firms reject more workers as more workers become available to them, and this feature gives rise to the monotonicity of our characterization map. While such monotonicity is known to admit a fixed point, possible indifferences in firms' preferences make it nontrivial to identify the exact forms of substitutable preferences required for existence.[7] We identify two different types of substitutable preferences with indifferences—a weak form leading to the existence of *a* stable matching and a strong form leading to the existence of *side-optimal* (i.e., *firm-optimal* and *worker-optimal*) stable matchings. We also identify a condition under which a side-optimal stable matching can be found via a generalized Gale–Shapley algorithm. Finally, we also find a condition, *richness*, that guarantees the uniqueness of the stable matching, thus generalizing the uniqueness result of Azevedo and Leshno (2016) beyond the special case of responsive preferences. When firms have responsive preferences but face general group-specific quotas (e.g., affirmative actions), our richness condition is implied by a full support assumption on firms' preferences, leading to a unique stable matching under that assumption.

We next draw implications of our results from a continuum economy for "nearby" large finite economies, assuming that each firm has a continuous utility function over the measure of workers it matches with. Specifically, we show that any large finite economy that is sufficiently close to our continuum economy (in terms of the distribution of worker types) admits a matching that is *approximately stable* in the sense that the incentives for blocking

[5]In the public school choice program in Boston prior to 2005, for instance, a student's priority at a school was based only on broad criteria, such as the student's area of residence and whether he or she had any siblings currently enrolled at that school. Consequently, at each school, many students were assigned the same priority (Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005)).

[6]For instance, it allows for Leontief-type preferences with respect to alternative types of workers, in which firms desire to hire each type of workers in equal size.

[7]If a firm's preferences are responsive, an arbitrary resolution of indifferences—or tie-breaking—preserves responsiveness and thus implies existence. For more general preferences, however, a random or arbitrary tie-breaking of indifferences does not necessarily lead to a choice function that possesses the necessary properties for existence.

are arbitrarily small. The converse also holds: namely, if any approximately stable matchings defined over a sequence of large finite economies converge to a matching in the limit continuum economy, then the limit matching constitutes an (exact) stable matching in the continuum economy. In addition, approximately stable matchings in large finite economies share similar structural properties (e.g., side-optimality and uniqueness) with the stable matchings in the continuum economy. Our results thus suggest the usefulness of the continuum economy as a tool for studying large finite economies.

Finally, we study the fairness and incentive properties of matching. Stability eliminates justified envy and thus protects workers from being discriminated by a firm against the workers it perceives as *less desirable*. However, stability alone is silent on the fairness of a matching in terms of treating workers who are perceived by a firm as *equivalent*. This issue is particularly relevant in school choice since schools evaluate students based on coarse priorities. Kesten and Ünver (2014) showed that, given responsive preferences by schools (i.e., firms in our model), it is possible to implement a matching that eliminates discrimination among students enjoying the same priority. We show that this stronger notion of fairness can be achieved *even with general preferences*, either in a continuum economy or in a finite but "time-share" model in which schools/firms and students/workers can share time or match probabilistically (see Sotomayor (1999), Alkan and Gale (2003), and Kesten and Ünver (2014), among others). In addition, the mechanism implementing such a strongly fair matching gives workers an incentive to truthfully reveal their preferences.

Relationship With the Literature

The present paper is connected with several strands of literature. Most importantly, it is related to the growing literature on matching and market design in the traditions of Gale and Shapley (1962) and Roth (1984). While substitutability has been recognized in the literature as crucial for the existence of stable matchings (Kelso and Crawford (1982), Roth (1985), Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), Hatfield and Kominers (2017)), our paper shows that substitutability is not necessary for the existence of a (approximately) stable matching when there are a large number of agents on one side of the market.

Our study was inspired by recent research on matching with a continuum of agents due to Abdulkadiroğlu, Che, and Yasuda (2015) and Azevedo and Leshno (2016).[8] As in the present study, these authors assumed that there are a finite number of firms and a continuum of workers. In particular, Azevedo and Leshno (2016) showed the existence and uniqueness of a stable matching in that setting. However, as opposed to the present study, these authors assumed that firms have responsive preferences—which is a special case of substitutability. Our contribution is to show that such restrictions are not necessary for the existence of a stable matching in the continuum economy.

An independent and contemporaneous study by Azevedo and Hatfield (2018) (henceforth, AH) also analyzed matching with a continuum of agents.[9] Consistent with our study,

[8]Various recent studies on large matching markets are also related but formally different, such as Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2009), Kojima and Manea (2010), Manea (2009), Che and Kojima (2010), Lee (2017), Liu and Pycia (2016), Che and Tercieux (2018a, 2018b), Ashlagi, Kanoria, and Leshno (2017), Miralles (2008), Miralles and Pycia (2017), Kojima, Pathak, and Roth (2013), and Hatfield, Kojima, and Narita (2016).

[9]Although not as closely related, our study is also analogous to Azevedo, Weyl, and White (2013), who demonstrated the existence of competitive equilibrium in an exchange economy with a continuum of agents and indivisible objects.

these authors found that a stable matching exists even when not all agents have substitutable preferences. However, the two studies have several notable differences. First, AH considered a continuum of firms each employing a finite number of workers; thus, they considered a continuum of agents on both sides of the market. By contrast, the present paper considers a finite number of firms that each employ a continuum of workers. These two models thus provide complementary approaches for studying large markets, and they are applicable to different environments.[10] Second, AH assumed that there is a finite number of both firm and worker types, which enables them to use Brouwer's fixed-point theorem to demonstrate the existence of a stable matching. By contrast, we place no restriction on the number of worker types and thus allow for both finite and infinite numbers of types, and this generality in type spaces requires a topological fixed-point theorem from functional analysis.[11]

Subsequent to the current work, Wu (2017) proved core existence in a class of games he labeled *convex matching games*. While this domain contains both AH's setup and a special case of our setup, there are at least two differences. First, his core notion uses "strong domination," that is, a blocking coalition is required to make *every agent in the block strictly better off*, while our blocking notion uses "weak domination," which allows indifferent agents to be part of a blocking coalition.[12] This difference makes our notion of stability stronger and not implied by his core existence result. Second, he assumed the finiteness of agent types, and this is crucial for his proof method that relies on Scarf's lemma (Scarf (1967)).

Our paper joins the growing literature that characterizes a stable matching via a fixed point of a suitably defined operator (see Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Ostrovsky (2008), and Hatfield and Kominers (2017), among others). This literature ensures monotonicity of fixed-point operators by assuming substitutable preferences and obtains existence by applying Tarski's fixed-point theorem. By contrast, a significant part of our paper does not impose substitutability restrictions on preferences; instead, we rely on the continuum of workers—along with continuity in firms' preferences—to guarantee the continuity of the operator (in an appropriately chosen topology). This approach allows us to use a generalization of the Kakutani fixed-point theorem, a more familiar tool in traditional economic theory that is used in existence proofs of general equilibrium and Nash equilibrium. Even for substitutable preferences, we are able to generalize the condition for existence and other properties of interest by accommodating indifferences.

The present paper is related to the literature on general equilibrium. With a finite number of consumers, the convexity of consumer preferences is key for establishing the existence of Walrasian equilibria (Arrow and Debreu (1954), McKenzie (1954, 1959)). With-

[10]For example, in the context of school choice, many school districts consist of a small number of schools that each admit hundreds of students, which fits well with our approach. However, in a large school district such as New York City, the number of schools is large compared with the number of students per school, and the AH model may offer a good approximation.

[11]To the best of our knowledge, this type of mathematics has never been applied to two-sided matching, and we view the introduction of these tools into the matching literature as one of our methodological contributions. Our model also has the advantage of subsuming Azevedo and Leshno (2016) and many other studies that assume a continuum of worker types. Moreover, the substantive issues studied in these papers are significantly different. Indifferences in preferences, substitutable preferences, incentives, and fairness are studied only by the present paper, while many-to-many matching, core, and general equilibrium are studied only by AH.

[12]In matching theory, weak domination is more standard. It appears to be more natural because, for instance, strong domination does not allow a firm to form a block by combining new workers with existing workers, whereas weak domination allows for such a block.

out the convexity assumption, Aumann (1966) showed the existence in a continuum economy, while Starr (1969) showed the existence of an approximate equilibrium in a large finite economy.[13] Especially close to our study are models with clubs, most notably Ellickson, Grodal, Scotchmer, and Zame (1999, 2001).[14] Similarly to our study, these papers consider both large finite and continuum economies and show the existence of (approximate) equilibria using Kakutani's fixed-point theorem. Despite these similarities, there are also a number of notable differences. First, Ellickson et al. (1999, 2001) assumed the existence of private goods and transfers, neither of which is assumed in our model. Moreover, in their model, the size of clubs (groups) and the number of agent types are finite. In this respect, their model is closer to AH's model in which a continuum of firms each hire a finite number of workers.

The current paper is also related to the literature on matching with couples. Like a firm in our model, a couple can be seen as a single agent with complementary preferences over contracts, and this complementarity may lead to non-existence (see Roth (1984) and Klaus and Klijn (2005)). The large market existence results with couples by Kojima, Pathak, and Roth (2013) and Ashlagi, Braverman, and Hassidim (2014) are of similar spirit to the current paper. However, "couples" are the only source of complementarities in these papers, and their results require the proportion of couples to be insignificant in the large market, an assumption we do not make here.[15]

The remainder of this paper is organized as follows. Section 2 presents an example to illustrate our main contributions. Section 3 describes a matching model in the continuum economy. Section 4 provides a fixed-point characterization of stable matchings in the continuum economy. Sections 5 and 6 provide the existence of a stable matching under continuous and substitutable preferences, respectively. In Section 7, we explore implications of our existence results for approximately stable matchings in large finite economies. In Section 8, we investigate fairness and strategy-proofness. Section 9 concludes.

2. ILLUSTRATIVE EXAMPLE

Before proceeding to our formal model, it is useful to illustrate the main issues and the idea of the paper using simple examples. These examples will also serve as a tool for explaining some technical concepts introduced in the model section.

We first consider a simple finite matching market to illustrate the non-existence problem caused by a complementary preference. Suppose that there are two firms, f_1 and f_2, and two workers, θ and θ'. The agents have the following preferences:

$$\theta : f_1 \succ f_2; \qquad f_1 : \{\theta, \theta'\} \succ \emptyset;$$
$$\theta' : f_2 \succ f_1; \qquad f_2 : \{\theta\} \succ \{\theta'\} \succ \emptyset.$$

In other words, worker θ prefers f_1 to f_2, and worker θ' prefers f_2 to f_1; firm f_1 prefers employing both workers to employing neither, which the firm in turn prefers to employing only one of the workers; and firm f_2 prefers worker θ to θ', whom it in turn prefers to

[13]See also a related result on an approximate core by Shapley and Shubik (1966).

[14]Although less close to our paper, other notable contributions include Ellickson (1979), Scotchmer and Wooders (1987), Gilles and Scotchmer (1997), and Scotchmer and Shannon (2015) as well as a survey by Sandler and Tschirhart (1997).

[15]Also see Pycia (2012) and Echenique and Yenmez (2007), who studied many-to-one matching with complementarities and peer effects, and Nguyen and Vohra (2018), who studied how one can minimally modify firms' quotas to guarantee a stable matching in a problem with couples. These papers allow for complementarities, but they do not study large economies.

employing neither. Firm f_1 has a "complementary" (or more precisely, non-substitutable) preference in the sense that availability of one worker causes it to demand the other. To illustrate how complementary preferences cause instability in this example, recall that stability requires that there be no blocking coalition. Due to f_1's complementary preference, it must employ either both workers or neither worker in any stable matching. In the former case, the matching is unstable since f_2 must be unmatched and can form a blocking coalition with type θ' worker, who prefers firm f_2 to firm f_1. In the latter case, the matching is also unstable since f_2 will hire only θ, which leaves θ' unemployed; this outcome will be blocked by f_1 forming a coalition with θ and θ' that will benefit all members of the coalition.

Can stability be restored if the market becomes large? If the market remains finite, the answer is no. To illustrate this proposition, consider a scaled-up version of the above model: there are q workers of type θ and q workers of type θ', and they have the same preferences as previously described. Firm f_2 prefers type-θ workers to type-θ' workers and wishes to hire in that order but at most a total of q workers. Firm f_1 has a complementary preference for hiring identical numbers of type-θ and type-θ' workers (with no capacity limit). Formally, if x and x' are the numbers of available workers of types θ and θ', respectively, then firm f_1 would choose $\min\{x, x'\}$ workers of each type.

When q is odd (including the original economy, where $q = 1$), a stable matching does not exist.[16] To illustrate this, first note that if firm f_1 hires more than $q/2$ workers of each type, then firm f_2 has a vacant position to form a blocking coalition with a type-θ' worker (who prefers f_2 to f_1). If f_1 hires fewer than $q/2$ workers of each type, then some workers will remain unmatched (because f_2 hires at most q workers). If a type-θ worker is unmatched, then f_2 will form a blocking coalition with that worker. If a type-θ' worker is unmatched, then firm f_1 will form a blocking coalition by hiring that worker and a θ worker (possibly matched with f_2).

Consequently, "exact" stability is not guaranteed, even in a large finite market. Nevertheless, approximate stability is achievable in the sense that the "magnitude" of instability diminishes as the economy grows large. To illustrate this, let q be odd and consider a matching:

$$M^q = \begin{pmatrix} f_1 & f_2 \\ \frac{q+1}{2}\theta + \frac{q+1}{2}\theta' & \frac{q-1}{2}\theta + \frac{q-1}{2}\theta' \end{pmatrix}, \tag{1}$$

where the notation here indicates that firms f_1 and f_2 are matched respectively to $\frac{q+1}{2}$ and $\frac{q-1}{2}$ workers of each type (we will use an analogous notation throughout). This matching is unstable because f_2 has one vacant position it wants to fill and there is a type-θ' worker who is matched to f_1 but prefers f_2. However, this scenario is the only possible block of this matching, and it involves only one worker. As the economy grows large, if one additional worker becomes insignificant for firm f_2 relative to its size, a property formalized later as continuous preferences, then the payoff consequence of forming such a block must also become insignificant. In this sense, the instability problem becomes insignificant.

This idea can be seen most clearly in the limit of the above economy. Suppose that there is a unit mass of workers, half of whom are type θ and the other half of whom are

[16]We sketch the argument here; Section S.1 of the Supplemental Material (Che, Kim, and Kojima (2019)) provides the argument in fuller form. When q is even, a matching in which each firm hires $\frac{q}{2}$ of each type of workers is stable.

type θ'. Their preferences are the same as described above. Suppose that firm f_1 wishes to maximize $\min\{x, x'\}$, where x and x' are the measures of type-θ and type-θ' workers, respectively. Firm f_2 can hire at most $\frac{1}{2}$ of the workers, and it prefers to fill as much of this quota as possible with type-θ workers and fill the remaining quota with type-θ' workers. In this economy, there is a (unique) stable matching in which each firm hires exactly one-half of the workers of each type:

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{4}\theta + \frac{1}{4}\theta' & \frac{1}{4}\theta + \frac{1}{4}\theta' \end{pmatrix}. \tag{2}$$

To see that this matching is stable, note that any blocking coalition involving firm f_1 requires taking away a positive—and identical—measure of type-θ' and type-θ workers from firm f_2, which is impossible because type-θ' workers will object to it. Additionally, any blocking coalition involving firm f_2 requires that a positive measure of type-θ workers be taken away from firm f_1 and supplant the same measure of type-θ' workers in f_2's workforce, which is impossible because type-θ workers will object to it. Our analysis below will demonstrate that the continuity of firms' preferences, which will be defined more precisely, is responsible for guaranteeing the existence of a stable matching in the continuum economy and approximate stability in the large finite economies in this example.

3. MODEL OF A CONTINUUM ECONOMY

Agents and Their Measures

There is a finite set $F = \{f_1, \dots, f_n\}$ of firms and a mass of workers. Let $\emptyset$ be the null firm, representing the possibility of not being matched with any firm, and define $\tilde{F} := F \cup \{\emptyset\}$. The workers are identified with types $\theta \in \Theta$, where Θ is a compact metric space with metric d^Θ. Let Σ denote a Borel σ-algebra of space Θ. Let $\overline{\mathcal{X}}$ be the set of all nonnegative measures such that, for any $X \in \overline{\mathcal{X}}$, $X(\Theta) \leq 1$. Assume that the entire population of workers is distributed according to a nonnegative (Borel) measure $G \in \overline{\mathcal{X}}$ on (Θ, Σ). In other words, for any $E \in \Sigma$, $G(E)$ is the measure of workers belonging to E. For normalization, assume that $G(\Theta) = 1$.

Any subset of the population or **subpopulation** is represented by a nonnegative measure X on (Θ, Σ) such that $X(E) \leq G(E)$ for all $E \in \Sigma$.[17] Let $\mathcal{X} \subset \overline{\mathcal{X}}$ denote the set of all subpopulations. We further say that a nonnegative measure $\tilde{X} \in \mathcal{X}$ is a **subpopulation of** $X \in \mathcal{X}$, denoted as $\tilde{X} \sqsubset X$, if $\tilde{X}(E) \leq X(E)$ for all $E \in \Sigma$. We let $\mathcal{X}_X$ denote the set of all subpopulations of X. Note that $(\mathcal{X}, \sqsubset)$ is a partially ordered set.[18] As usual, for any two subpopulations (or measures) $X, Y \in \mathcal{X}$, $X + Y$ and $X - Y$ denote their sum and difference, respectively.

Given the partial order $\sqsubset$, for any $X, Y \in \mathcal{X}$, we define $X \vee Y$ (join) and $X \wedge Y$ (meet) to be the supremum and infimum of X and Y, respectively.[19] Additionally, for

[17] In the case of finitely many types, we will use "measure" and "mass" interchangeably.

[18] The reflexivity, transitivity, and antisymmetry of the order are easy to check.

[19] For instance, $X \vee Y$ is the smallest measure of which both X and Y are subpopulations. It can be shown that for all $E \in \Sigma$,

$$(X \vee Y)(E) = \sup_{D \in \Sigma} X(E \cap D) + Y(E \cap D^c).$$

any $\mathcal{X}' \subset \mathcal{X}$, let $\bigvee \mathcal{X}'$ and $\bigwedge \mathcal{X}'$ denote the supremum and infimum of $\mathcal{X}'$, which are well defined since the partially ordered set $(\mathcal{X}, \sqsubset)$ is a complete lattice, as we show in Section S.2.1 of the Supplemental Material.

EXAMPLE 1—Leading Example: Consider the example from the previous section, henceforth called the leading example. Its limit economy is a continuum economy with $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\{\theta\}) = G(\{\theta'\}) = 1/2$.[20] Let $X = (x, x')$ and $Y = (y, y')$ be two measures in our leading example, where x and x' are the measures of types θ and θ', respectively, under X, and likewise y and y' under Y. Then, their join and meet are measures $X \vee Y = (\max\{x, y\}, \max\{x', y'\})$ and $X \wedge Y = (\min\{x, y\}, \min\{x', y'\})$, respectively.

EXAMPLE 2—Interval Economy: Consider a continuum economy with type space $\Theta = [0, 1]$ and suppose the measure G admits a bounded density g for all $\theta \in [0, 1]$. In this case, it easily follows that for $X, Y \sqsubset G$, their densities x and y are well defined.[21] Then, their join $Z = (X \vee Y)$ and meet $Z' = (X \wedge Y)$ admit densities z and z' defined by $z(\theta) = \max\{x(\theta), y(\theta)\}$ and $z'(\theta) = \min\{x(\theta), y(\theta)\}$ for all θ, respectively.

Consider the space of all (signed) measures (of bounded variation) on (Θ, Σ). We endow this space with a weak-$*$ topology and its subspace $\mathcal{X}$ with the relative topology. Given a sequence of measures (X_k) and a measure X on (Θ, Σ), we write $X_k \xrightarrow{w^*} X$ to indicate that (X_k) converges to X as $k \to \infty$ under weak-$*$ topology and simply say that (X_k) **weakly converges** to X.[22]

Agents' Preferences

We now describe agents' preferences. Each worker is assumed to have a strict preference over $\tilde{F}$. Let a bijection $P : \{1, \dots, n+1\} \to \tilde{F}$ denote a worker's preference, where $P(j)$ denotes the identity of the worker's j^{th} best alternative, and let $\mathcal{P}$ denote the (finite) set of all possible worker preferences.

We write $f \succ_P f'$ to indicate that f is strictly preferred to f', according to P. (We sometimes write $f \succ_\theta f'$ to express the preference of a particular type θ.) For each $P \in \mathcal{P}$, let $\Theta_P \subset \Theta$ denote the set of all worker types whose preference is given by P, and assume that Θ_P is measurable and $G(\partial\Theta_P) = 0$, where $\partial\Theta_P$ denotes the boundary of Θ_P.[23] Because all worker types have strict preferences, Θ can be partitioned into the sets in $\mathcal{P}_\Theta := \{\Theta_P : P \in \mathcal{P}\}$.

We next describe firms' preferences. We do so indirectly by defining a firm f's **choice correspondence** $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$, where $C_f(X) \subset \mathcal{X}_X$ is a nonempty set of subpopulations of

[20]Henceforth, given any measure X, $X(\theta)$ will denote a measure of the singleton set $\{\theta\}$ to simplify notation.

[21]For instance, X is Lipschitz continuous, and thus its density is well defined, since $|X([0, \theta']) - X([0, \theta])| \le |G([0, \theta']) - G([0, \theta])| \le \bar{g}|\theta' - \theta|$, where $\bar{g} := \sup_s g(s)$.

[22]We use the term "weak convergence" because it is common in statistics and mathematics, although weak-$*$ convergence is a more appropriate term from the perspective of functional analysis. As is well known, $X_k \xrightarrow{w^*} X$ if $\int_\Theta h\, dX_k \to \int_\Theta h\, dX$ for all bounded continuous functions h. See Theorem 12 in Appendix A for some implications of this convergence.

[23]This is a technical assumption that facilitates our analysis. The assumption is satisfied if, for each $P \in \mathcal{P}$, Θ_P is an open set such that $G(\bigcup_{P \in \mathcal{P}} \Theta_P) = G(\Theta)$: all agents, except for a measure-zero set, have strict preferences, a standard assumption in the matching theory literature. The assumption that $G(\partial\Theta_P) = 0$ is also satisfied if Θ is discrete. To see it, note that $\partial E := \overline{E} \cap \overline{E^c}$, where $\overline{E}$ and $\overline{E^c}$ are the closures of E and E^c, respectively. Then, we have $\overline{E} = E$ and $\overline{E^c} = E^c$, so $\overline{E} \cap \overline{E^c} = E \cap E^c = \emptyset$. Hence, the assumption is satisfied.

X for any $X \in \mathcal{X}$.[24] We assume that C_f satisfies the **revealed preference property**: for any $X, X' \in \mathcal{X}$ with $X \sqsubset X'$, if $C_f(X') \cap \mathcal{X}_X \neq \emptyset$, then $C_f(X) = C_f(X') \cap \mathcal{X}_X$.[25] Let $R_f : \mathcal{X} \rightrightarrows \mathcal{X}$ be a **rejection correspondence** defined by $R_f(X) := \{Y \in \mathcal{X} | Y = X - X' \text{ for some } X' \in C_f(X)\}$. By convention, we let $C_\emptyset(X) = \{X\}, \forall X \in \mathcal{X}$, meaning that $R_\emptyset(X)(E) = 0$ for all $X \in \mathcal{X}$ and $E \in \Sigma$. We will call C_f (resp,. R_f) a firm f's **choice** (resp., **rejection**) **function** if $|C_f(X)| = 1$ for all $X \in \mathcal{X}$. In this case, we slightly abuse the notation to write a unique outcome of the function without the set notation.

In our leading example, the choice correspondences of firms f_1 and f_2 are functions respectively given by

$$\begin{aligned} &C_{f_1}(x_1, x_1') = (\min\{x_1, x_1'\}, \min\{x_1, x_1'\}) \quad \text{and} \\ &C_{f_2}(x_2, x_2') = \left(x_2, \min\left\{\frac{1}{2} - x_2, x_2'\right\}\right), \end{aligned} \tag{3}$$

when $x_i \in [0, \frac{1}{2}]$ of type-θ workers and $x_i' \in [0, \frac{1}{2}]$ of type-θ' workers are available to firm f_i, $i = 1, 2$.

In sum, a continuum economy is summarized as a tuple $\Gamma = (G, F, \mathcal{P}_\Theta, C_F)$.

Matching and Stability

A **matching** is $M = (M_f)_{f \in \tilde{F}}$ such that $M_f \in \mathcal{X}$ for all $f \in \tilde{F}$ and $\sum_{f \in \tilde{F}} M_f = G$. Firms' choice correspondences can be used to define a binary relation describing firms' preferences over matchings. For any two matchings, M and M', we say that firm f prefers M_f' to M_f if $M_f' \in C_f(M_f' \vee M_f)$, and write $M_f' \succeq_f M_f$.[26] We also say that f strictly prefers M_f' to M_f if $M_f' \succeq_f M_f$ holds while $M_f \succeq_f M_f'$ does not, and write $M_f' \succ_f M_f$. The resulting preference relation amounts to taking a minimal stance on the firms' preferences, limiting attention to those revealed via their choices. Given this preference relation, we denote $M' \succeq_F M$ if $M_f' \succeq_f M_f$ for all $f \in F$, and $M' \succ_F M$ if $M' \succeq_F M$ and $M_f' \succ_f M_f$ for some $f \in F$.

To discuss workers' welfare, fix any matching M and any firm f. Let

$$D^{\succeq f}(M) := \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F} : f' \succeq_P f} M_{f'}(\Theta_P \cap \cdot) \quad \text{and} \quad D^{\preceq f}(M) := \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F} : f' \preceq_P f} M_{f'}(\Theta_P \cap \cdot) \tag{4}$$

denote the measure of workers assigned to firm f or better (according to their preferences) and the measure of workers assigned to firm f or worse (again, according to their preferences), respectively, where $M_{f'}(\Theta_P \cap \cdot)$ denotes a measure that takes the value $M_{f'}(\Theta_P \cap E)$ for each $E \in \Sigma$. Starting from M as a default matching, the latter measures the number of workers who would rather match with f. Meanwhile, the former measure is useful for characterizing the workers' overall welfare. For any two matchings M and

[24] Taking firms' choices as a primitive offers flexibility with regard to the preferences over alternatives that are not chosen. This approach is also adopted by other studies in matching theory, which include Alkan and Gale (2003) and Aygün and Sönmez (2013), among others.

[25] This property must hold if the choice is made by a firm optimizing with a well-defined preference relation. The property is often invoked in the matching theory literature (see Hatfield and Milgrom (2005), Fleiner (2003), and Alkan and Gale (2003)). Recently, Aygün and Sönmez (2013) clarified the role of this property in the context of matching with contracts.

[26] This relation is known as the *Blair order* in the literature (see Blair (1984)).

M', we say that $M' \succeq_\Theta M$ if $D^{\succeq f}(M) \sqsubset D^{\succeq f}(M'), \forall f \in \tilde{F}$ and $M' \succ_\Theta M$ if $M' \succeq_\Theta M$ and $D^{\succeq f}(M) \neq D^{\succeq f}(M')$ for some $f \in \tilde{F}$.[27] In other words, for each firm f, if the measure of workers assigned to f or better is larger in one matching than in the other, then we can say that the workers' overall welfare is higher in the former matching.

Equipped with these notions, we can define stability.

DEFINITION 1: A matching M is **stable** if

1. (Individual Rationality) For each $f \in F$, $M_f \in C_f(M_f)$; for each $P \in \mathcal{P}$, $M_f(\Theta_P) = 0, \forall f \prec_P \emptyset$; and
2. (No Blocking Coalition) No $f \in F$ and $M'_f \in \mathcal{X}$ exist such that $M'_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$.

Condition 1 requires that no firm wish to unilaterally drop any of its matched workers and that each matched worker prefer being matched to being unmatched.[28] Condition 1 requires that there be no firm and no set of workers who are not matched together but prefer to be. When Condition 1 is violated by f and M'_f, we say that f and M'_f **block** M.

Two notions closely related to stability are *group stability* and *Pareto efficiency*. Group stability requires that no group of firms and workers gain from blocking a matching. Pareto efficiency requires that a matching not be Pareto-dominated or not blocked by the all-inclusive coalition. As is standard in many-to-one matching, stability is equivalent to group stability and implies Pareto efficiency. The formal statements and proofs are given in Sections S.2.2 and S.2.3 of the Supplemental Material (Che, Kim, and Kojima (2019)).

4. A CHARACTERIZATION OF STABLE MATCHING

This section characterizes stable matchings, which will serve as a tool for establishing their existence in the subsequent sections. Stability exhausts the blocking opportunities for all firms, which requires each firm to choose optimally from the workers "available" to that firm. Hence, to identify a stable matching, one must identify *the set of workers available to each firm*. This set is inherently of a fixed-point character: the availability of a worker to a firm depends on the set of firms willing to match with her, but that set depends in turn on firms' optimization, given the workers "available" to them.

The preceding logic suggests that a stable matching is associated with a fixed point of a mapping—or, more intuitively, a stationary point of a process that repeatedly revises the set of available workers to the firms based on the preferences of the workers and the firms. Formally, we define a map $T : \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}^{n+1}$ such that, for each $X \in \mathcal{X}^{n+1}$,

$$T(X) := \Bigg\{ \tilde{X} \in \mathcal{X}^{n+1} \;\Bigg|\; \text{there exists } (Y_f)_{f\in\tilde{F}} \text{ with } Y_f \in R_f(X_f), \forall f \in \tilde{F}, \text{ such that} \\ \tilde{X}_f(\cdot) = \sum_{P:P(1)=f} G(\Theta_P \cap \cdot) + \sum_{P:P(1)\neq f} Y_{f_-^P}(\Theta_P \cap \cdot), \forall f \in \tilde{F} \Bigg\}, \tag{5}$$

[27] Note that this comparison is made in the aggregate matching sense without keeping track of the identities of workers who get better off with M'.

[28] We note that the first part of Condition 1 (namely, $M_f \in C_f(M_f)$ for each $f \in F$) is implied by Condition 1. To see this, suppose $M_f \notin C_f(M_f)$. Let $M'_f \in C_f(M_f)$. Then, since $M'_f \sqsubset M_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$, Condition 1 is violated. We opted to write that condition to follow the convention in the literature and ease the exposition.

where $f_-^P \in \tilde{F}$, called the **immediate predecessor** of f at P, is a firm that is ranked immediately above firm f according to P.[29] This mapping takes a profile X of available workers as input and returns a nonempty set of profiles of available workers. For each $X \in \mathcal{X}^{n+1}$, $T(X)$ is nonempty because $R_f(X)$ is nonempty for each $f \in \tilde{F}$ by assumption.

To explain, fix a firm f. Consider first the worker types Θ_P who rank f as their first-best choice (i.e., $f = P(1)$). All such workers are available to f, which explains the first term of (5). Consider next the worker types Θ_P who rank f as their second-best choice (i.e., $f = P(2)$). Within this group, only the workers rejected by their top-choice firm $P(1) = f_-^P$ are available to f, which explains the second term of (5). Now, consider the worker types Θ_P who rank f as their third-best choice (i.e., $f = P(3)$). Within this group, only the workers rejected by *both* their first- and second-choice firms, that is, $P(1)$ and $P(2)$, would be available to f. To calculate the measure of these workers, however, one may focus on those available to and rejected by $P(2) = f_-^P$ since, by the previous observation, the workers available to $P(2)$ are those who were rejected by $P(1)$. The analogous explanation applies to all firms on workers' rank order lists.

The map T can be interpreted as a tâtonnement process in which an auctioneer iteratively quotes firms' "budgets" (in terms of the measures of available workers). As in a classical Walrasian auction, the budget quotes are revised based on the preferences of the market participants, reducing the budget for firm f when more workers are demanded by the firms ranked above f and increasing the budget otherwise. Once the process converges, one reaches a fixed point, having found the workers who are "truly" available to firms—those who are compatible with the preferences of all market participants.

REMARK 1: The mapping T can be seen as mimicking Gale and Shapley's deferred acceptance algorithm (DA), particularly the worker-proposing one. To see this, consider the case in which each C_f is a choice function. Then, we can write T as a profile $(T_f)_{f\in\tilde{F}}$, where, for each $X \in \mathcal{X}^{n+1}$,

$$T_f(X) = \sum_{P:P(1)=f} G(\Theta_P \cap \cdot) + \sum_{P:P(1)\neq f} R_{f_-^P}(X_{f_-^P})(\Theta_P \cap \cdot). \tag{6}$$

For each firm f, this mapping returns the workers who are rejected by an immediate predecessor of f. These are analogous to the workers who propose to firm f in the worker-proposing DA algorithm, since they are those rejected by the immediate predecessor. Indeed, this analogy becomes precise when the firms' preferences are substitutable (i.e., when each R_f is monotonic): each iteration of the mapping T (starting from zero subpopulations) coincides with the cumulative measures of workers proposing at a corresponding step of worker-proposing DA. This result is shown in Section S.3 of the Supplemental Material. Our fixed-point mapping resembles those developed in the context of finite matching markets (e.g., see Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006)), but the construction here differs since a continuum of workers draw their types from a very rich space and are treated in aggregate terms without being distinguished by their identities.

We now present our main characterization theorem.

THEOREM 1: *There exists a stable matching M with $X_f = D^{\preceq f}(M), \forall f \in \tilde{F}$ if and only if $(X_f)_{f\in\tilde{F}}$ is a fixed point of T (i.e., $X \in T(X)$).*

[29]Formally, $f_-^P \succ_P f$ and $f' \succeq_P f_-^P$ for any $f' \succ_P f$.

PROOF: See Appendix A. *Q.E.D.*

This characterization identifies the measures X of workers available to firms in a stable matching as a fixed point of T. A stable matching $M = (M_f)_f$ is then obtained as firms' optimal choices from X that satisfy $X_f = D^{\preceq f}(M)$ for each $f \in \tilde{F}$.[30] This process is illustrated in the next example.

EXAMPLE 3—Leading Example: Consider Example 1 again. The candidate measures of available workers are denoted by a tuple $X = (X_{f_1}, X_{f_2}) = (x_1, x_1', x_2, x_2') \in [0, \frac{1}{2}]^4$, where $X_{f_i} = (x_i, x_i')$ is the measures of type-θ and type-θ' workers available to f_i. Since f_1 and f_2 are the top choices for θ and θ', respectively, all these workers are available to the respective firms according to our T mapping. Thus, without loss we can set $x_1 = G(\theta) = \frac{1}{2}$ and $x_2' = G(\theta') = \frac{1}{2}$ and consider $(\frac{1}{2}, x_1', x_2, \frac{1}{2})$ as our candidate measures. The firms' choice functions are then given by (3) while the fixed-point mapping in (6) is given by $T = (T_{f_1}, T_{f_2})$, where

$$T_{f_1}(X) := \left(\frac{1}{2}, R_{f_2}\left(x_2, \frac{1}{2}\right)(\theta')\right) = \left(\frac{1}{2}, x_2\right), \tag{7}$$

$$T_{f_2}(X) := \left(R_{f_1}\left(\frac{1}{2}, x_1'\right)(\theta), \frac{1}{2}\right) = \left(\frac{1}{2} - x_1', \frac{1}{2}\right). \tag{8}$$

Thus, $(x_1, x_1'; x_2, x_2')$ is a fixed point of T if and only if $(x_1, x_1'; x_2, x_2') = (\frac{1}{2}, x_2; \frac{1}{2} - x_1', \frac{1}{2})$, or $x_1 = x_2' = \frac{1}{2}$ and $x_1' = x_2 = \frac{1}{4}$. Each firm's optimal choice from the fixed point—that is, $C_{f_i}(X_{f_i})$ for each $i = 1, 2$—then gives a (unique) stable matching M defined in (2).

In light of Theorem 1, the existence of a stable matching reduces to the existence of a fixed point of T. The next two sections identify two sufficient conditions for the latter.

5. THE EXISTENCE OF A STABLE MATCHING IN THE CONTINUUM ECONOMY

Based on our characterization result, we now present the main existence result under the standard continuity assumption on firms' choice correspondences. We say that firm f's choice correspondence C_f is **upper hemicontinuous** if, for any sequences $(X^k)_{k\in\mathbb{N}}$ and $(\tilde{X}^k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ such that $X^k \xrightarrow{w^*} X$, $\tilde{X}^k \xrightarrow{w^*} \tilde{X}$, and $\tilde{X}^k \in C_f(X^k), \forall k$, we have $\tilde{X} \in C_f(X)$.[31] As suggested by the name, upper hemicontinuity means that a firm's choice changes continuously with the distribution of available workers. We say that C_f is **convex-valued** if $C_f(X)$ is a convex set for any $X \in \mathcal{X}$.[32]

DEFINITION 2: Firm $f \in F$ has a **continuous preference** if C_f is upper hemicontinuous and convex-valued.

[30]Importantly, an arbitrary selection from $C_f(X_f)$ for each $f \in F$ at the fixed point X need not lead to a matching, let alone a stable one. Care is needed to construct a stable matching. Equation (14) in Appendix A provides a precise formula to obtain a stable matching M from a fixed point X of T. We thank a referee for raising a question that led us to clarify this issue.

[31]This definition is often referred to as the "closed graph property," which implies (the standard definition of) upper hemicontinuity and closed-valuedness if the range space is compact, as is true in our case.

[32]By the familiar observation based on Berge's maximum theorem (see Ok (2011) for instance), an upper hemicontinuous and convex-valued choice correspondence arises when a firm has a utility function $u : \mathcal{X} \to \mathbb{R}$ that is continuous (in weak-$*$ topology) and quasi-concave.

Many complementary preferences are compatible with continuous preferences. Recall Example 3, for instance, in which firm f_1 has a Leontief-type preference: it wishes to hire an equal number of workers of types θ and θ' (specifically, the firm wants to hire type-θ workers only if type-θ' workers are also available, and vice versa). As Example 3 shows, a stable matching exists despite the extreme complementarity. Also note that firms' preferences are clearly continuous; this is not a mere coincidence, as we now show that continuity of firms' preferences implies the existence of a stable matching:

THEOREM 2: *If each firm $f \in F$ has a continuous preference, then a stable matching exists.*

PROOF: See Appendix A. *Q.E.D.*

Given the fixed-point characterization of stable matchings in Theorem 1, our proof approach is to show that T has a fixed point. To this end, we first demonstrate that the upper hemicontinuity of firm preferences implies that the mapping T is also upper hemicontinuous. We also verify that $\mathcal{X}$ is a compact and convex set. Upper hemicontinuity of T and compactness and convexity of $\mathcal{X}$ allow us to apply the Kakutani–Fan–Glicksberg fixed-point theorem to guarantee that T has a fixed point.[33] Then, the existence of a stable matching follows from Theorem 1, which shows the equivalence between the set of stable matchings and the set of fixed points of T.

Although the continuity assumption is quite general, including preferences not allowed for in the existing literature, it is not without a restriction, as we illustrate next.

EXAMPLE 4—Role of Upper Hemicontinuity: Consider the following economy modified from Example 3: There are two firms f_1 and f_2, and two worker types, θ and θ', each with measure $1/2$. Firm f_1 wishes to hire *exactly* measure $1/2$ of each type and prefers to be unmatched otherwise. Firm f_2's preference is responsive subject to the capacity of measure $1/2$: it prefers type-θ to type-θ' workers and prefers the latter to leaving a position vacant. It follows that C_{f_1} violates upper hemicontinuity, while C_{f_2} does not. As before, we assume

$$\theta : f_1 \succ f_2;$$
$$\theta' : f_2 \succ f_1.$$

No stable matching exists in this environment, as shown in Section S.4 of the Supplemental Material.

The upper hemicontinuity assumption is important for the existence of a stable matching; this example shows that non-existence can occur even if the choice function of only one firm violates upper hemicontinuity. This example also suggests that non-existence can reemerge when some "lumpiness" is reintroduced into the continuum economy (i.e., one firm can only hire a minimum mass of workers). However, this kind of lumpiness may not be very natural in a continuum economy, which is unlike a finite economy where lumpiness is a natural consequence of the indivisibility of each worker.

By comparison, the convex-valuedness may rule out some realistic situations:[34]

[33]For the Kakutani–Fan–Glicksberg fixed-point theorem, refer to Theorem 16.12 and Corollary 16.51 in Aliprantis and Border (2006).

[34]In a classical competitive market context, Farrell (1959) studied how specialization and indivisibility can lead to non-convexity and discussed its implications for the existence of equilibria.

EXAMPLE 5—Role of Convex-Valuedness: Let us again modify Example 3 as follows. The preferences of f_1 and of the two worker types remain the same, while the masses of type-θ and type-θ' workers are 0.6 and 0.4, respectively. Firm f_2 specializes in only one type of workers and prefers hiring as many workers as possible: If x and x' are the available masses of the two types, then the firm only hires mass x of type θ if $x > x'$ and only mass x' of type θ' if $x < x'$, but never desires to mix the two types. If $x = x'$, the firm is indifferent between hiring either type of mass x (again without mixing types). It is straightforward to verify that the choice correspondence corresponding to this preference is upper hemicontinuous. However, it is not convex-valued since, for any $x = x' > 0$, the firm's choice set contains $(x, 0)$ and $(0, x)$ but not any (strict) convex combination of them. Consequently, a stable matching does not exist in this case (see Section S.4 of the Supplemental Material).

REMARK 2—Algorithm to Find a Fixed Point of T: It will be useful to have an algorithm to find or at least approximate a stable matching, which is equivalent to approximating a fixed point of T. One such algorithm is the *tâtonnement process*, that is, to apply T iteratively starting from an initial point $X^0 \in \mathcal{X}^{n+1}$. Unfortunately, this algorithm does not always work. To see this, consider the mapping T in (7) and (8), and let $\phi_1(x_2) := R_{f_2}(x_2, \frac{1}{2})(\theta') = x_2$ and $\phi_2(x'_1) := R_{f_1}(\frac{1}{2}, x'_1)(\theta) = \frac{1}{2} - x'_1$. Then, T is effectively reduced to a mapping: $(x'_1, x_2) \mapsto (\phi_1(x_2), \phi_2(x'_1))$, which is depicted as in Figure 1(a). While its fixed point exists (i.e., the intersection in Figure 1(a)), if one starts anywhere else, say a point X^0 in that figure, the algorithm gets trapped in a cycle.

The map T could work for other situations, however. For instance, modify Example 3 yet again so that the firm f_1 would like to hire mass $a < 1$ of type-θ workers per unit mass of type-θ' workers. Then, the mapping (ϕ_1, ϕ_2) changes to the one in Figure 1(b), where the tâtonnement process converges to a unique fixed point irrespective of the starting point; see Figure 1(b).[35] In fact, the composite map $T^2 = T \circ T$ in this modified example is a contraction mapping, so the convergence result can be understood by invoking the following generalized version of the contraction mapping theorem (see Chapter 3 of Ok (2017) for instance):

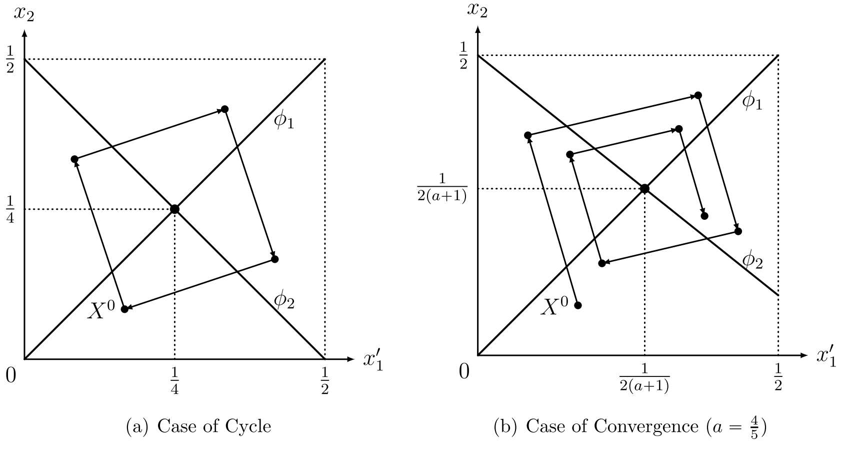

FIGURE 1.—Fixed point of mapping T.

[35] See Section S.4 of the Supplemental Material for a detailed analysis.

PROPOSITION 1: *Suppose that T is singleton-valued and let $\tilde{T} = T^m$ denote a function obtained from iterating T by m times. If $\tilde{T}$ is a contraction mapping, then, starting with any $X^0 \in \mathcal{X}^{n+1}$, $X^k := \tilde{T}(X^{k-1})$ converges to a unique fixed point of T as $k \to \infty$.*

While the contraction mapping theorem provides a condition for our mapping T to serve as an algorithm for finding its fixed point, it need not be the only condition. We will later see another convergence result when firms have substitutable preferences (see Part (ii) of Theorem 4).

6. SUBSTITUTABLE PREFERENCES

In this section, we study another class of preferences known as substitutable preferences in the framework of a continuum economy. Although substitutable preferences have been studied extensively, there are at least three reasons to study them in our context. First, substitutable preferences yield useful results beyond existence, such as side-optimal stable matchings and a constructive algorithm, and it is interesting to see if these results generalize to a large market. Further, as will be seen, substitutable preferences need not be continuous, so the existence of a stable matching is not implied by Theorem 2. Second, most existing studies on substitutable preferences are confined to the domain of strict preferences.[36] However, indifferences are a prevalent feature of many markets (see, for instance, Abdulkadiroğlu, Pathak, and Roth (2009)), and yet little is known regarding whether existence and other useful properties such as side-optimal stable matchings hold in the weak preference domain.[37] Third, the large market setting raises another important question—uniqueness. Azevedo and Leshno (2016) offered sufficient conditions for a stable matching to be unique in the large economy but in the restricted domain of "responsive" preferences. It is interesting to ask whether uniqueness extends to general substitutable preferences.

6.1. *Existence and Side-Optimality*

To define substitutable preferences in our general domain, we need a few definitions. Given a partial order $\sqsubset$, a correspondence $h : \mathcal{X} \rightrightarrows \mathcal{X}$ is said to be *weak-set monotonic* if it satisfies the following: (i) for any $X \sqsubset X'$ and $Z \in h(X)$, there is $Z' \in h(X')$ with $Z \sqsubset Z'$; (ii) for any $X \sqsubset X'$ and $Z' \in h(X')$, there is $Z \in h(X)$ with $Z \sqsubset Z'$.[38]

DEFINITION 3: Firm f's preference is **weakly substitutable** if R_f is weak-set monotonic.

The current definition preserves the well-known property of a firm becoming more selective as more workers are available. The novelty here is that substitutability is defined for a rejection correspondence (instead of a rejection function, as in the literature). Indeed, the definition can be seen as a generalization of the standard notion: if $C_f(X)$ is

[36] Sotomayor (1999) is a notable exception.

[37] Existence for general substitutable preferences is not clear, unlike the case of responsive preferences. In the latter case, an arbitrary tie-breaking (e.g., random tie-breaking) preserves responsiveness, leading to existence. To our knowledge, there is no straightforward generalization of this method to the general class of substitutable preferences.

[38] The weak-set monotonicity is weaker than the strong-set monotonicity often used in monotone comparative statics (e.g., Milgrom and Shannon (1994)).

singleton-valued for all $X \in \mathcal{X}$, this notion collapses to the requirement that R_f be monotonic in the underlying order $\sqsubset$: $R_f(X) \sqsubset R_f(X')$ whenever $X \sqsubset X'$.

We now establish the existence result in the domain of weakly substitutable preferences:

THEOREM 3: *If each firm's preference is weakly substitutable and each C_f is closed-valued, then a stable matching exists.*[39]

PROOF: See Appendix B. *Q.E.D.*

As before, this result rests on the existence of a fixed point of the correspondence T defined earlier. One can see that if firms have weakly substitutable preferences, then T is weak-set monotonic. While Zhou (1994) extended Tarski's well-known theorem to the case of correspondences, his monotonicity condition is stronger than ours, so we instead apply a recent result due to Li (2014) to prove the existence of a fixed point. The weakening of the required condition is important in that it allows us to accommodate indifferences that arise naturally: for instance, consider a firm with a fixed quota that can be filled with any mixture of multiple types, as featured in the next example.

EXAMPLE 6—Weak Substitutability: Suppose that there are three firms, f_1, f_2, and f_3, and two worker types, θ and θ', and that the capacity of each firm and the mass of each worker type are all equal to $\frac{1}{2}$. The workers' preferences are

$$\theta : f_1 \succ f_2 \succ f_3;$$
$$\theta' : f_1 \succ f_3 \succ f_2.$$

Firms f_2 and f_3 have responsive preferences: they both prefer θ to θ' (i.e., they wish to hire in that order up to the capacity of $\frac{1}{2}$). Firm f_1 is indifferent between the two types of workers, and its preference is described by a choice correspondence:

$$C_{f_1}(x, x') = \left\{ (y, y') \in [0, x] \times [0, x'] | y + y' = \min\left\{ x + x', \frac{1}{2} \right\} \right\}.$$

This choice correspondence satisfies weak substitutability, as one can easily check. There exists a continuum of stable matchings:[40] for any $z \in [0, \frac{1}{2}]$, it is a stable matching for firm f_1 to hire mass z of type-θ workers and $\frac{1}{2} - z$ of type-θ' workers, for firm f_2 to hire mass $\frac{1}{2} - z$ of type-θ workers, and for firm f_3 to hire mass z of type-θ' workers. Clearly, as z increases, firm f_2 becomes worse off and firm f_3 becomes better off. Hence, the firm-optimal stable matching does not exist, and neither does the worker-optimal stable

[39]The closed-valuedness is a mild condition that may hold even if the choice correspondence fails to be upper hemicontinuous, as demonstrated by the example in footnote 40.

[40]In this example, firms' preferences satisfy the conditions of Theorem 2, so Theorem 3 is not needed for showing the existence of a stable matching. However, one can easily obtain an example in which the latter theorem applies while the former does not. In Example 6, suppose that firm f_1 is instead endowed with a choice correspondence defined as follows: for some $\bar{x} \in [0, 1/2]$,

$$C_{f_1}(x, x') = \begin{cases} \{(x, x')\} & \text{if } x' < \bar{x}, \\ \{(0, y') | y' \in [\bar{x}, x']\} & \text{if } x' \geq \bar{x}. \end{cases}$$

This correspondence fails to be upper hemicontinuous, rendering Theorem 2 inapplicable, but the conditions of Theorem 3 are satisfied, as can be checked easily.

matching since firm f_1 hires type-θ and type-θ' workers in different proportions across different stable matchings.[41]

We next introduce a stronger notion of substitutability that would restore side-optimality. We say a set $\mathcal{X}' \subset \mathcal{X}$ of subpopulations is a *complete sublattice* if $\mathcal{X}'$ contains both $\bigvee \mathcal{Z}$ and $\bigwedge \mathcal{Z}$ for every set $\mathcal{Z} \subset \mathcal{X}'$.[42]

DEFINITION 4: Firm f's preference is **substitutable** if (i) R_f is weak-set monotonic and (ii) for any $X \in \mathcal{X}$, $R_f(X)$ is a complete sublattice.[43]

When C_f is singleton-valued, the condition reduces to the standard notion of substitutability, so the distinction between the two different versions of substitutability disappears. Nevertheless, the requirements for substitutable preferences are stronger in the current weak preference domain. In particular, (ii) is a strong requirement that preferences such as those described by C_{f_1} in Example 6 fail.[44]

At the same time, substitutable preferences do accommodate certain types of indifferences. Imagine, for instance, a school that has a selective program with a limited quota and a general program with flexible quotas. For the selective program, the school admits students in the order of their scores up to its quota. Once the quota is reached, the school may admit students for the general program with flexible quotas and without consideration of their scores. To our knowledge, the next result is the first to establish the existence of side-optimal stable matchings in the weak preference domain.[45]

THEOREM 4: *Suppose that each firm's preference is substitutable. Then, the following results hold: letting $\mathcal{M}^*$ denote the set of stable matchings,*

(i) (Side-Optimal Stable Matching) *There exist stable matchings, $\overline{M}, \underline{M} \in \mathcal{M}^*$, that are firm-optimal/worker-pessimal and firm-pessimal/worker-optimal, respectively, in the following senses: If $M \in \mathcal{M}^*$, then $M \succeq_\Theta \overline{M} \succeq_F M$ and $M \preceq_\Theta \underline{M} \preceq_F M$.*

(ii) (Generalized Gale–Shapley) *If, in addition, C_f is order-continuous for each f,[46] then the limit of the algorithm that iteratively applies $\overline{T}$ starting with $X_f = G, \forall f \in \tilde{F}$, produces a firm-optimal stable matching, and the limit of the algorithm that iteratively applies $\underline{T}$ starting with $X_f = \mathbf{0}, \forall f \in \tilde{F}$, produces a worker-optimal stable matching, where $\overline{T}(X) := \bigvee T(X)$ and $\underline{T}(X) := \bigwedge T(X)$ for any $X \in \mathcal{X}^{n+1}$.*

[41]Both firm-optimal and worker-optimal stable matchings are defined in Theorem 4(i).

[42]Authors use different terminologies for the same property: Topkis (1998) called it *subcomplete sublattice* and Zhou (1994) called it *closed sublattice*.

[43]This condition is weaker than Zhou (1994)'s which requires *strong-set monotonicity* in place of (i). Our substitutability guarantees side-optimality but not a complete lattice, which Zhou's condition guarantees. See Example S1 in Section S.5 of the Supplemental Material for the case in which our substitutability condition holds while the strong-set monotonicity fails, causing the lattice structure to fail.

[44]To see this, note $\mathcal{Z} = \{(\frac{1}{2}, 0), (0, \frac{1}{2})\} \subset R_{f_1}(\frac{1}{2}, \frac{1}{2})$, but $\bigvee \mathcal{Z} = (\frac{1}{2}, \frac{1}{2}) \notin R_{f_1}(\frac{1}{2}, \frac{1}{2})$, so R_{f_1} is not a sublattice (let alone a complete one).

[45]Theorem 4 does not require closed-valuedness of the choice correspondences, which Theorem 3 requires. It is often the case, however, that Part (ii) of the substitutability (i.e., the complete sublattice property) implies closed-valuedness. For instance, the relation holds if there are finitely many worker types so $\mathcal{X}$ is a subset of a finite-dimensional Euclidean space.

[46]A correspondence C is *order-continuous* if $\overline{C}(X_k) \xrightarrow{w^*} \overline{C}(X)$ for any increasing sequence $X_k \xrightarrow{w^*} X$, and $\underline{C}(X_k) \xrightarrow{w^*} \underline{C}(X)$ for any decreasing sequence $\mathcal{X}_k \xrightarrow{w^*} X$, where $\overline{C}(X) = \bigvee C(X)$ and $\underline{C}(X) = \bigwedge C(X)$ for any $X \in \mathcal{X}$.

PROOF: See Appendix B. *Q.E.D.*

While the existence of firm-optimal and worker-optimal stable matchings is well known for the strict preference domain, no such result is previously known for the weak preference domain. In fact, the received wisdom is that firms' indifferences are incompatible with the presence of side-optimal stable matchings even in a more restrictive domain such as responsive preferences. Theorems 3 and 4, taken together with Example 6, clarify the types of indifferences that permit the existence of side-optimal stable matchings and those that do not. In particular, responsive preferences with indifferences (studied by Abdulkadiroğlu, Pathak, and Roth (2009) and Erdil and Ergin (2008), for instance) satisfy weak substitutability but fail substitutability and, consistent with Theorems 3 and 4, guarantee the existence of a stable matching but not a side-optimal one.

The second part of Theorem 4 shows that a generalized version of Gale–Shapley's deferred acceptance algorithm finds a side-optimal stable matching but only with the additional (order-) continuity assumption.[47] Without this continuity property, the algorithm may get "stuck" at an unstable matching (Example S2 in Section S.5 of the Supplemental Material illustrates this point).

Next, we adapt another well-known condition to our context:

DEFINITION 5: Firm f's preference exhibits the **law of aggregate demand (or LoAD)** if, for any $X, X' \in \mathcal{X}$ with $X \sqsubset X'$, $\sup C_f(X)(\Theta) \leq \inf C_f(X')(\Theta)$.[48]

Given LoAD and substitutability, we show that the total measure of workers employed by each firm in any stable matching is uniquely pinned down:

THEOREM 5—Rural Hospital: *If each firm's preference is substitutable and satisfies LoAD, then, for any $M \in \mathcal{M}^*$, we have $M_f(\Theta) = \overline{M}_f(\Theta)$, $\forall f \in F$ and $M_\emptyset = \overline{M}_\emptyset$.*

PROOF: See Appendix B. *Q.E.D.*

REMARK 3—Finite Economy: While the results are established for our continuum economy model, they apply to finite economy models with little modification. For instance, the order-continuity required for Theorem 4(ii) would be satisfied vacuously in the finite economy. To the extent that these results were obtained in the extant literature for strict preferences, the current results would amount to their extensions to more general preferences in the finite economy context.

6.2. *Uniqueness of Stable Matching*

Azevedo and Leshno (2016) established the uniqueness of a stable matching in a continuum economy when firms have responsive preferences. We now investigate the extent to which the uniqueness result extends to the general substitutable preferences environment. The uniqueness question is important not only for the continuum economy but also

[47]This result is reminiscent of the well-known property of a supermodular game whereby, given the order-continuity property, iterative deletion of strictly dominated strategies starting from the "largest" and "smallest" strategies produces the largest and smallest Nash equilibria, respectively. See Milgrom and Roberts (1990) and Milgrom and Shannon (1994).

[48]This property is an adaptation of a property that appears in the literature, including Hatfield and Milgrom (2005), Alkan (2002), and Fleiner (2003).

for the large finite one, as will be shown in the next section. Expanding the domain beyond responsive preferences helps to identify the underlying condition that drives uniqueness.

In this section, we assume that each firm's choice is not only substitutable but also unique, that is, each C_f is a choice function. The substitutability condition guarantees existence of the worker-optimal stable matching $\underline{M}$. For any matching M, firm f, and subset F' of firms, we let $M^f_{F'}$ be a subpopulation of workers defined by

$$M^f_{F'}(E) := \sum_{P\in\mathcal{P}} \sum_{f':f\succ_P f', f'\notin F'} M_{f'}(\Theta_P \cap E) \quad \text{for each } E \in \Sigma.$$

In words, this is the measure of workers who are matched outside firms F' and available to firm f under M (excluding those matched with f).[49] Consider the following property:

DEFINITION 6—Rich Preferences: The preferences are **rich** if, for any individually rational matching $\hat{M} \neq \underline{M}$ such that $\hat{M} \succeq_F \underline{M}$, there exists $f^* \in F$ such that $\underline{M}_{f^*} \neq C_{f^*}((\underline{M}_{f^*} + \hat{M}^{f^*}_{\bar{F}}) \wedge G)$, where $\bar{F} := \{f \in F | \hat{M}_f \succ_f \underline{M}_f\}$.

The condition is explained as follows. Consider any individually rational matching $\hat{M}$ that is preferred to the worker-optimal stable matching $\underline{M}$ by all firms and strictly so by firms in $\bar{F} \subset F$. Then, the richness condition requires that at matching $\underline{M}$, there must exist a firm f^* that would be happy to match with some workers who are not hired by the firms in $\bar{F}$ and are willing to match with f^* under $\hat{M}$. Since firms are more selective at $\hat{M}$ than at $\underline{M}$, it is intuitive that in the latter matching, a firm would demand some workers whom the more selective firms would not demand in the former matching. The presence of such worker types requires richness in the preference palettes of firms and workers—hence the name. This point will be seen more clearly in the next section when one considers (a general class of) responsive preferences.

THEOREM 6: *Suppose that each firm's preference is substitutable and satisfies LoAD. If the preferences are rich, then a unique stable matching exists.*

PROOF: See Appendix B. *Q.E.D.*

Both richness and substitutability are necessary for the uniqueness result, as one can construct counterexamples without much difficulty. LoAD is also indispensable for the uniqueness, as demonstrated by Example S3 in the Supplemental Material. (Recall that the LoAD is trivially satisfied by the responsive preferences of Azevedo and Leshno (2016).)

While rich preferences may not be easy to check, one can show that the condition is implied by a full-support condition in a general class of environments that nests Azevedo and Leshno (2016) as a special case, as demonstrated below.

Responsive Preferences With Submodular Quotas

Suppose that firms have responsive preferences but may face quotas on the number of workers they can hire from different groups of workers. Such group-specific quotas,

[49]Note that this is a valid subpopulation, or a measure, since it is the sum of a finite number of measures.

which are typically based on socio-economic status or other characteristics, may arise from affirmative action or diversity considerations. The resulting preferences (or choice functions) may violate responsiveness but nonetheless satisfy substitutability.

Assume that there is a finite set $\mathcal{T}$ of "ethnic types" that describe worker characteristics such as ethnicity, gender, and socio-economic status, such that type θ is assigned an ethnic type $\tau(\theta)$ via some measurable function $\tau : \Theta \to \mathcal{T}$. For each $t \in \mathcal{T}$, a (measurable) set $\Theta^t := \{\theta \in \Theta | \tau(\theta) = t\}$ of agents has an ethnic type t. Each firm f faces a quota constraint given by function $\mathcal{Q}_f : 2^{\mathcal{T}} \to \mathbb{R}_+$ such that, for each $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{Q}_f(\mathcal{T}')$ is a maximum quota (in terms of the measure of workers) the firm f can hire from the ethnic types in $\mathcal{T}'$. We assume that $\mathcal{Q}_f(\emptyset) = 0$, $\mathcal{Q}_f(\mathcal{T}) > 0$, and $\mathcal{Q}_f$ is **submodular**: for any $\mathcal{T}', \mathcal{T}'' \subset \mathcal{T}$,

$$\mathcal{Q}_f(\mathcal{T}') + \mathcal{Q}_f(\mathcal{T}'') \geq \mathcal{Q}_f(\mathcal{T}' \cup \mathcal{T}'') + \mathcal{Q}_f(\mathcal{T}' \cap \mathcal{T}'').$$

Submodularity allows for the most general form of group-specific quotas that encompasses all existing models: for instance, it holds if the firm faces arbitrary quotas on a *hierarchical family* of subsets of $\mathcal{T}$.[50] This case includes a familiar case studied by many authors (Abdulkadiroğlu and Sönmez (2003), for instance) in which the family forms a partition of $\mathcal{T}$. Subject to the quotas, each firm has *responsive preferences* given by a (measurable) score function $s_f : \Theta \to [0, 1]$ such that f prefers type-θ to type-θ' workers if and only if $s_f(\theta) > s_f(\theta')$. For simplicity, we assume that no positive mass of types has an identical score.[51]

Clearly, this class of preferences subsumes pure responsive preferences considered by Azevedo and Leshno (2016) as a special case, but includes preferences that fail their condition. We can show that these preferences satisfy both substitutability and LoAD:

LEMMA 1: *A firm f with responsive preferences facing submodular quotas exhibits a choice function that satisfies substitutability and LoAD.*[52]

PROOF: See Section S.6.2 of the Supplemental Material. *Q.E.D.*

Specifically, Section S.6 of the Supplemental Material provides an algorithm that finds the choice function for a firm with this type of preference and shows that the choice function satisfies substitutability and LoAD. Given the prevalence of group-specific constraints, this lemma, which is highly nontrivial, may be of interest in its own right. For instance, since the choice of each firm is a function, substitutability implies that the set of stable matchings has a lattice structure, a conclusion that does not hold under general choice correspondence, even with substitutability.

Next, we generalize the full-support condition of Azevedo and Leshno (2016) to the current setup:

DEFINITION 7: The worker population has a **full support** if, for each preference $P \in \mathcal{P}$, any ethnic type $t \in T$, and for any nonempty open cube set $S \subset [0, 1]^n$, the worker types

$$\Theta^t_P(S) := \{\theta \in \Theta_P \cap \Theta^t | (s_f(\theta))_{f \in F} \in S\}$$

have a positive measure, that is, $G(\Theta^t_P(S)) > 0$.

[50]A family of sets is hierarchical if, for any sets $\mathcal{T}', \mathcal{T}''$, either $\mathcal{T}' \cap \mathcal{T}'' = \emptyset$, $\mathcal{T}' \subset \mathcal{T}''$, or $\mathcal{T}'' \subset \mathcal{T}'$. See Che, Kim, and Mierendorff (2013) for the proof of this result.

[51]This assumption is maintained by Azevedo and Leshno (2016), for instance.

[52]Section S.6.4 of the Supplemental Material presents an example in which the substitutability fails due to the quota constraints, which are not submodular.

Note that this condition boils down to that of Azevedo and Leshno (2016) if $\mathcal{T}$ is a singleton set.

PROPOSITION 2: *Suppose that each firm $f \in F$ has responsive preferences and faces submodular quotas. Then, the full-support condition implies the richness condition.*

PROOF: See Section S.6.3 of the Supplemental Material. *Q.E.D.*

Combining Lemma 1, Proposition 2, and Theorem 6, we conclude as follows:

COROLLARY 1: *Suppose that each firm $f \in F$ has responsive preferences and faces submodular quotas. If the full-support condition holds, then a unique stable matching exists.*

Since the model of Azevedo and Leshno (2016) constitutes a special case of submodular quotas, their uniqueness result in Theorem 1-1 is implied by this corollary.

7. APPROXIMATE STABILITY IN FINITE ECONOMIES

In Section 2, we observed that a finite economy, however large, may not possess a stable matching, while a large finite economy admits a matching that is stable in an approximate sense. Motivated by this observation and building on our findings in the continuum economy, we formalize here the notion of approximate stability and demonstrate that the set of approximately stable matchings in large finite economies inherits the desirable properties of stable matching in a "nearby" continuum economy. Specifically, the set is nonempty, contains (approximately) firm-optimal and worker-optimal matchings, and consists of virtually unique matching whenever the corresponding property is true for the continuum economy. These results suggest that a continuum economy provides a good framework for analyzing large finite economies, which is useful since a continuum economy often permits a more tractable analysis, as demonstrated by Azevedo and Leshno (2016).

To analyze economies of finite sizes, we consider a sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ indexed by the total number of workers $q \in \mathbb{N}$. In each economy Γ^q, there is a fixed set of n firms, $f_1, \dots, f_n$, that does not vary with q. As before, each worker has a type in Θ. The worker distribution is normalized with the economy's size. Formally, let the (normalized) population G^q of workers in Γ^q be defined so that $G^q(E)$ represents the number of workers with types in E divided by q. A (discrete) measure X^q is **feasible** in economy Γ^q if $X^q \sqsubset G^q$, and it is a measure whose value for any E is a multiple of $1/q$. Let $\mathcal{X}^q$ denote the set of all feasible subpopulations in Γ^q. Note that G^q, and thus every $X^q \in \mathcal{X}^q$, belongs to $\overline{\mathcal{X}}$, although it need not be a subpopulation of G and therefore may not belong to $\mathcal{X}$. Let us say that a sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ **converges to a continuum economy** Γ if $G^q \xrightarrow{w^*} G$.

To formalize approximate stability, we first represent each firm f's preference by a cardinal utility function $u_f : \overline{\mathcal{X}} \to \mathbb{R}$ defined over normalized distributions of workers it matches with. This utility function represents a firm's preference for each finite economy Γ^q as well as for the continuum economy Γ.[53] We assume that u_f is continuous in

[53]The assumption that the same utility function applies to both finite and limit economies is made for convenience. The results in this section hold if, for instance, the utilities in finite economies converge uniformly to the utility in the continuum economy.

weak-$*$ topology.[54] Then, firm f chooses a feasible subpopulation that maximizes u_f in the respective economies. Specifically, in the continuum economy Γ, the firm's choice correspondence is given by

$$C_f(X) = \operatorname*{arg\,max}_{X' \sqsubset X} u_f(X'), \quad \forall X \in \mathcal{X}. \tag{9}$$

In each finite economy Γ^q, it is given by

$$C_f^q(X) := \operatorname*{arg\,max}_{X' \sqsubset X, X' \in \mathcal{X}^q} u_f(X'), \quad \forall X \in \mathcal{X}^q. \tag{10}$$

All our results in this section rely on the existence of a stable matching in the continuum economy, which holds if each u_f is such that C_f defined in (9) satisfies the conditions in Theorem 2 or in Theorem 3. For instance, the conditions in Theorem 2 are satisfied if each u_f is quasi-concave in addition to being continuous, since C_f is then convex-valued and upper hemicontinuous.[55]

A **matching** in finite economy Γ^q is $M^q = (M_f^q)_{f \in \tilde{F}}$ such that $M_f^q \in \mathcal{X}^q$ for all $f \in \tilde{F}$ and $\sum_{f \in \tilde{F}} M_f^q = G^q$. The measure of available workers for each firm f at matching $M^q \in (\mathcal{X}^q)^{n+1}$ is $D^{\preceq f}(M^q)$, where $D^{\preceq f}(\cdot)$ is defined as in (4).[56] Note that because each M_f^q is a multiple of $1/q$, $D^{\preceq f}(M^q)$ is feasible in Γ^q. We now define ϵ-stability in the finite economy Γ^q.

DEFINITION 8: For any $\epsilon > 0$, a matching $M^q \in (\mathcal{X}^q)^{n+1}$ is **ϵ-stable** in economy Γ^q if (i) for each $f \in F$, $M_f^q \in C_f^q(M_f^q)$; (ii) for each $P \in \mathcal{P}$, $M_f^q(\Theta_P) = 0, \forall f \prec_P \emptyset$; and (iii) $u_f(\tilde{M}^q) < u_f(M_f^q) + \epsilon$ for any $f \in F$ and $\tilde{M}^q \in \mathcal{X}^q$ with $\tilde{M}^q \sqsubset D^{\preceq f}(M^q)$.[57]

Conditions (i) and (ii) of this definition are analogous to the corresponding conditions for exact stability, so ϵ-stability relaxes stability only with respect to Condition (iii). Specifically, an ϵ-stable matching could be blocked, but if so, the gain from blocking must be small for any firm.[58] An ϵ-stable matching will be robust against blocks if a rematching

[54] To guarantee the existence of such a utility function, we may assume, as in Remark 24, that each firm is endowed with a complete, continuous preference relation. Then, because the set of alternatives $\overline{\mathcal{X}}$ is a compact metric space, this preference can be represented by a continuous utility function according to the Debreu representation theorem (Debreu (1954)).

[55] The upper hemicontinuity is an implication of Berge's maximum theorem.

[56] To be precise, $D^{\preceq f}(M^q)$ is given as in (4) with G and M being replaced by G^q and M^q, respectively.

[57] Approximate stability might be defined slightly differently. Say a matching M^q is **ϵ-distance stable** if (i) and (ii) of Definition 8 hold and (iii') $d(\tilde{M}_f^q, M_f^q) < \epsilon$ for any coalition f and $\tilde{M}_f^q \in \mathcal{X}^q$ that blocks M^q in the sense that $\tilde{M}_f^q \sqsubset D^{\preceq f}(M^q)$ and $u_f(\tilde{M}_f^q) > u_f(M_f^q)$, where $d(\cdot, \cdot)$ is the Lévy–Prokhorov metric (which metrizes the weak-$*$ topology). In other words, if a matching M^q is ϵ-distance stable, then the distance of any alternative matching a firm proposes for blocking must be within ϵ from the original matching. One advantage of this concept is that it is ordinal, that is, we need not endow the firms with cardinal utility functions to formalize the notion. Note that the notion also requires the ϵ bound for *any* blocking coalition, not just the "optimal" blocking coalition as defined in Definition 1-1, making the notion of ϵ-distance stability more robust. In Section S.7.2 of the Supplemental Material, we prove the existence of ϵ-distance stable matching (under an additional mild assumption).

[58] Note that the Conditions (i) and (iii) are asymmetric in the sense that the matching should be precisely optimal against the blocking by an individual firm alone and only approximately optimal against the blocking by a coalition. We adopt this asymmetry because blocking with workers outside the firm is presumably more difficult for a firm to implement than retaining or firing its own workers.

process requires cost (at least of ϵ) for the firm to initiate a block, which seems sensible when there are some frictions in the market.

Our main result of this section follows:

THEOREM 7: *Suppose a sequence of economies* $(\Gamma^q)_{q\in\mathbb{N}}$ *converges to a continuum economy* Γ *which admits a stable matching* M. *Then, for any* $\epsilon > 0$, *there exists* $Q \in \mathbb{N}$ *such that, for all* $q > Q$, *there is an* ϵ*-stable matching* M^q *in* Γ^q.[59]

PROOF: See Appendix C. *Q.E.D.*

This result implies that a large finite market admits an approximately stable matching even with non-substitutable preferences. Interestingly, a converse of Theorem 7 also holds:

THEOREM 8: *Suppose a sequence of matchings* $(M^q)_{q\in\mathbb{N}}$ *converges to* M *and has the property that, for every* $\epsilon > 0$, *there exists* $Q \in \mathbb{N}$ *such that, for all* $q > Q$, M^q *is* ϵ*-stable in* Γ^q. *Then,* M *is stable in* Γ.[60]

PROOF: See Appendix C. *Q.E.D.*

This result implies that the behavior of large finite economies is well approximated by the continuum economy in the sense that by studying the latter, we will not "miss" any approximately stable matching in the former.

EXAMPLE 7: Recall the finite economy in Section 2, where there are q workers of each type.[61] Recall its limit economy admits a unique stable matching $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. If the index q is odd, then a stable matching does not exist. As we have already seen, the matching M^q defined in (1) is ϵ-stable in Γ^q for sufficiently large q and converges to the (unique) stable matching in Γ.[62] Also, as Theorem 8 indicates, any ϵ-stable matching in Γ^q for sufficiently large q must be close to the stable matching M in Γ defined in (2). For instance, any matching that is bounded away from M will be subject to a block that increases either firm's utility by more than a small ϵ.

The approximately stable matching established in Theorem 7 can be shown to possess other properties inherited from the structure of stable matchings in the continuum economy. To this end, we relax the notion of side-optimality.

[59]We note that M^q need not converge to M. In fact, there can be a stable matching in Γ that does not have any nearby approximate stable matching in large finite economy Γ^q (refer to Section S.7.3 of the Supplemental Material for an example), meaning that the (approximately) stable matching correspondence is not "lower hemicontinuous." This is the case because the exact individual rationality, that is, Condition (i) of Definition 8, can make a firm's choice in the finite economy never close to a certain stable matching in the continuum economy. If this condition is relaxed analogously to Condition (iii), then any stable matching in the continuum economy can be approximated by ϵ-stable matchings in large finite economies.

[60]This result is reminiscent of the upper hemicontinuity of Nash equilibrium correspondence (see, for instance, Fudenberg and Tirole (1991)). However, Theorem 8 establishes a more robust result in the sense that the convergence occurs even for "approximately" stable matchings in nearby economies.

[61]With a slight abuse of notation, this example assumes that there are a total of $2q$ workers (q workers of θ and θ' each) rather than q. Of course, this assumption is made for purely expositional purposes.

[62]This matching is also ϵ-distance stable since the only profitable block involves f_2 taking a single worker of type θ' away from firm f_1.

DEFINITION 9: For $\epsilon > 0$, a matching M^q is an **ϵ-firm-optimal stable matching** in Γ^q if there is $\delta \in (0, \epsilon)$ such that
1. M^q is δ-stable in Γ^q, and
2. for any matching $\hat{M}^q$ that is δ-stable in Γ^q, $u_f(M_f^q) \geq u_f(\hat{M}_f^q) - \epsilon, \forall f \in F$.

DEFINITION 10: For $\epsilon > 0$, a matching M^q is an **ϵ-worker-optimal stable matching** in Γ^q if there is $\delta \in (0, \epsilon)$ such that
1. M^q is δ-stable in Γ^q, and
2. for any matching $\hat{M}^q$ that is δ-stable in Γ^q,

$$D^{\succeq_f}\big(M^q\big)\big(E^\epsilon\big) \geq D^{\succeq_f}\big(\hat{M}^q\big)(E) - \epsilon, \quad \forall f \in F, \forall E \in \Sigma,$$

where $E^\epsilon := \{\theta \in \Theta | \exists \theta' \in E \text{ such that } d^\Theta(\theta, \theta') < \epsilon\}$ is the ϵ-neighborhood of E.

The ϵ-firm-optimality requires that the matching itself be approximately stable and that there be no other approximately stable matchings that make any firm better off by more than ϵ. The ϵ-worker-optimality can be seen as a natural extension of worker optimality, that is, $M^q \preceq_\Theta \hat{M}^q$, for the concept collapses to the latter if $\epsilon = 0$. We now prove the existence of approximately side-optimal matchings in large finite economies.[63]

THEOREM 9: *Suppose that a sequence of finite economies $(\Gamma^q)_{q\in\mathbb{N}}$ converges to a continuum economy Γ. Fix any $\epsilon > 0$.*

(i) *If there is a firm-optimal stable matching in Γ, then there is $Q \in \mathbb{N}$ such that, for all $q > Q$, an ϵ-firm-optimal stable matching in Γ^q exists.*

(ii) *If there is a worker-optimal stable matching $\underline{M}$ in Γ and $C_f(\underline{M}_f) = \{\underline{M}_f\}, \forall f \in F$ (i.e., for each firm f, $\underline{M}_f$ is its unique choice at $\underline{M}$), then there is $Q \in \mathbb{N}$ such that, for all $q > Q$, an ϵ-worker-optimal stable matching in Γ^q exists.*[64]

PROOF: See Appendix C. *Q.E.D.*

Finally, we show that if there is a unique stable matching in the limit economy Γ, then the approximately stable matching is *virtually unique* in any sufficiently large finite economy.

THEOREM 10: *Suppose that a sequence of finite economies $(\Gamma^q)_{q\in\mathbb{N}}$ converges to a continuum economy Γ which has a unique stable matching M. Then, the approximately stable matching of a large finite economy is "virtually unique" in the following sense: for any $\epsilon > 0$,*

[63]This result will be particularly useful when preferences are substitutable in a continuum economy but not in finite economies that converge to that economy. Delacrétaz, Kominers, and Teytelboym (2016) offered one such example in their study of refugee resettlement. Translated into our setup, there are three worker types, θ, θ', and θ'', and a firm f with capacity κ (or κ units of seats), which has a responsive preference with $\theta \succ \theta' \succ \theta''$. Each worker of types θ and θ'' occupies one seat, while a type-θ' worker occupies two seats. As Delacrétaz, Kominers, and Teytelboym (2016) showed, firm f's preference is not substitutable in finite economies, which is largely a result of the integer problem that disappears in a continuum economy. To see it, suppose that a continuum of workers $X = (x, x', x'')$ is available. Then, firm f's choice function is given by $C_f(X)(\theta) = \min\{x, \kappa\}$, $C_f(X)(\theta') = \min\{x', \frac{\kappa - C_f(X)(\theta)}{2}\}$, and $C_f(X)(\theta'') = \min\{x'', \kappa - C_f(X)(\theta) - 2C_f(X)(\theta')\}$. It is straightforward to check that this choice function represents a substitutable preference.

[64]Section S.7.3 of the Supplemental Material presents an example in which the result does not hold without the extra assumption, $C_f(\underline{M}_f) = \{\underline{M}_f\}, \forall f \in \tilde{F}$.

there are $Q \in \mathbb{N}$ *and* $\delta \in (0, \epsilon)$ *such that, for every* $q > Q$ *and for every* δ*-stable matching* $\hat{M}^q$ *in* Γ^q, *we have* $d(M, \hat{M}^q) < \epsilon$.[65]

PROOF: See Appendix C. *Q.E.D.*

This result, together with Theorem 6, leads to the following generalization of the convergence result (Theorem 2) in Azevedo and Leshno (2016).

COROLLARY 2: *Suppose that in the continuum economy* Γ, *firm preferences are substitutable and satisfy LoAD, and that the preferences are rich. Then, the approximately stable matching of any large finite economy* Γ^q *that converges to* Γ *is virtually unique.*

8. STRONG STABILITY AND STRATEGY-PROOFNESS

Stability promotes fairness by eliminating justified envy for workers. However, stability alone may not guarantee fair treatment of workers if a firm is indifferent over worker types that are unobservable or regarded as indistinguishable by the firm. The following example illustrates this point.

EXAMPLE 8: There are two firms f_1 and f_2, and a unit mass of workers of the following types:

$$\theta : f_1 \succ f_2 \succ \emptyset;$$
$$\theta' : f_2 \succ f_1 \succ \emptyset;$$
$$\theta'' : f_2 \succ \emptyset \succ f_1.$$

The type distribution is given by $G(\theta) = 1/2$ and $G(\theta') = 1/4 = G(\theta'')$. (Note that this example is the same as our leading example except that a mass of 1/4 of type-θ' workers now have a new preference θ''.)

Both firms are indifferent between type-θ' and type-θ'' workers; they differ only in their own preferences for firms. Firm f_1 wishes to maximize $\min\{x, x' + x''\}$, where x, x', and x'' are the measures of workers with types θ, θ', and θ'', respectively. Firm f_2 has a responsive preference with a capacity of 1/2 and prefers type θ to type θ' or θ''.

Consider first a mechanism that maps G to matching

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{4}\theta + \frac{1}{4}\theta' & \frac{1}{4}\theta + \frac{1}{4}\theta'' \end{pmatrix}.$$

This matching is stable, which can be seen by the fact that the firms are matched with the same measures of productivity types as in the stable matching in Example 3. Observe, however, that this matching treats the type-θ' and type-θ'' workers differently—the former workers match with f_1 and the latter workers match with f_2 (which they both prefer)—despite the fact that the firms perceive them as equivalent. This lack of "fairness" leads to an incentive problem: type-θ' workers have an incentive to (mis)report their type as θ'' and thereby match with f_2 instead of f_1.

[65]This finding implies that all stable matchings in any sufficiently large finite economy are also close to one another.

These problems can be addressed by another mechanism that maps G to a matching

$$\bar{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{6}\theta + \frac{1}{6}\theta' & \frac{1}{3}\theta + \frac{1}{12}\theta' + \frac{1}{12}\theta'' \end{pmatrix}.$$

Like M, this matching is stable, but in addition, firm f_2 treats type-θ' and type-θ'' workers identically in this matching. Further, neither type-θ' nor type-θ'' workers have incentives to misreport.

The fairness issue illustrated in this example is particularly relevant for school choice because schools evaluate students based on coarse priorities. Fairness demands that students who enjoy the same priorities be treated equally without any discrimination. This requirement calls for what Kesten and Ünver (2014) labeled *strong stability*, a condition satisfied by the second matching in the above example. As illustrated, strong stability is closely related to strategy-proofness for workers in a large economy. We thus address both issues here.

8.1. *Strong Stability and Strategy-Proofness in a Large Economy*

We begin by adapting our model to address the issues at hand. First, we denote each worker's type as a pair $\theta = (a, P)$, where a denotes the worker's productivity or skill and P describes her preferences over firms and the outside option, as above. We assume that worker preferences do not affect firm preferences and are private information, whereas productivity types may affect firm preferences and are observable to the firms (and to the mechanism designer). Let A and $\mathcal{P}$ be the sets of productivity and preference types, respectively, and $\Theta = A \times \mathcal{P}$. We assume that A is a finite set, which implies that Θ is a finite set, so the population G of worker types is a discrete measure.[66] We continue to assume that there is a continuum of workers.

The firms' preferences are also adapted for our environment. For each firm $f \in F$, worker types Θ are partitioned into $\mathbb{P}_f := \{\Theta_f^1, \dots, \Theta_f^{K_f}\}$ such that f is indifferent across all types within each indifference class $\Theta_f^k \subset \Theta$, for $k \in I_f := \{1, \dots, K_f\}$. Since a firm differentiates workers based only on their productivity types, we require that if $(a, P) \in \Theta_f^k$ for some $P \in \mathbb{P}_f$, then $(a, P') \in \Theta_f^k$ for all $P' \in \mathbb{P}_f$. At the same time, a firm can be indifferent across multiple productivity types in ways that are arbitrary and may differ across firms. We assume that each firm has a unique optimal choice in terms of the measure of workers in each indifference class, and let $\Lambda_f^k : \mathcal{X} \to \mathbb{R}_+$ denote firm f's unique choice of the total measure of workers in each indifference class Θ_f^k, $k \in I_f$,[67] which induces a

[66]The finiteness of A is necessitated by our use of weak-$*$ topology and the construction of strong stability and strategy-proof mechanisms below. To illustrate the difficulty caused by infinite A, suppose that A is a unit interval and G has a well-defined density. Our construction below would require that the density associated with firms' choice mappings satisfy a certain population proportionality property. Convergence in our weak-$*$ topology does not preserve this restriction on density. Consequently, the operator T may violate upper hemicontinuity, which may result in the failure of the nonempty-valuedness of our solution. It may be possible to address this issue by strengthening the topology, but whether the resulting space satisfies the conditions that would guarantee the existence of a stable matching remains an open question.

[67]Specifically, we assume that for each $X \sqsubset G$, $\Lambda_f^k(X) \in [0, \sum_{\theta \in \Theta_f^k} X(\theta)]$ and $\Lambda_f^k(X') = \Lambda_f^k(X)$ whenever $\sum_{\theta \in \Theta_f^{k'}} X'(\theta) = \sum_{\theta \in \Theta_f^{k'}} X(\theta)$ for all $k' \in I_f$. We also assume that $\Lambda_f^k(X') = \Lambda_f^k(X)$ whenever $\Lambda_f^{k'}(X) \leq \sum_{\theta \in \Theta_f^{k'}} X'(\theta) \leq \sum_{\theta \in \Theta_f^{k'}} X(\theta)$ for all $k' \in I_f$, which captures the revealed preference property.

choice correspondence

$$C_f(X) = \left\{ Y \sqsubset X \;\middle|\; \sum_{\theta \in \Theta_f^k} Y(\theta) = \Lambda_f^k(X), \forall k \in I_f \right\} \tag{11}$$

for each $X \in \mathcal{X}$. Continuity and substitutability of preferences can be defined in terms of Λ_f^k. If $\Lambda_f^k(\cdot)$ is continuous for each $k \in I_f$ (in the Euclidean topology), then the induced correspondence C_f is upper hemicontinuous and convex-valued. In that case, we simply say that a firm f's preference is **continuous**. Another case of interest is when $\sum_{\theta \in \Theta_f^k} X(\theta) - \Lambda_f^k(X)$ is nondecreasing in $(X(\theta))_{\theta \in \Theta}$ for each $k \in I_f$. In this case, the induced correspondence C_f is weakly substitutable, and we simply call a firm f's preference **weakly substitutable**.

As before, a matching is described by a profile $M = (M_f)_{f \in \tilde{F}}$ of subpopulations of workers matched with alternative firms or the null firm. We assume that all workers of the same (reported) type are treated identically ex ante. Hence, given matching M, a worker of type (a, P) in the support of G is matched to $f \in \tilde{F}$ with probability $\frac{M_f(a,P)}{G(a,P)}$. Note that $\sum_{f \in \tilde{F}} \frac{M_f(a,P)}{G(a,P)} = 1$ holds by construction, giving rise to a valid probability distribution over $\tilde{F}$. A **mechanism** is a function φ that maps any $G \in \overline{\mathcal{X}}$ to a matching.

We now introduce a strong notion of stability proposed by Kesten and Ünver (2014):

DEFINITION 11: A matching M is **strongly stable** if (i) it is stable and (ii) for any $f \in F$, $k \in I_f$, and $\theta, \theta' \in \Theta_f^k$, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$, then $\sum_{f' \in \tilde{F}: f' \prec_\theta f} M_{f'}(\theta) = 0$.

In words, strong stability requires that if a worker of type θ is assigned a firm f with strictly lower probability than another type θ' in the same indifference class for firm f, then the type-θ worker should never be assigned any firm f' that the worker ranks below f. In that sense, discrimination among workers in the same priority class should not occur.

Strategy-proofness can be defined via a stochastic dominance order, as proposed by Bogomolnaia and Moulin (2001).

DEFINITION 12: A mechanism φ is **strategy-proof for workers** if, for each (reported) population $G \in \overline{\mathcal{X}}$, productivity type $a \in A$, and preference types P and P' in $\mathcal{P}$ such that both (a, P) and (a, P') are in the support of G, and $f \in \tilde{F}$, we have

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geq \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)\big(a, P'\big)}{G\big(a, P'\big)}. \tag{12}$$

In words, strategy-proofness means that truthful reporting induces a random assignment for each worker that first-order stochastically dominates any random assignment that would result from untruthful reporting. Note that a worker can misreport only her preference type and not her productivity type (recall that a worker's productivity type determines firms' preferences regarding her).[68]

[68]Note also that unlike in finite population models, the worker cannot alter the population G by unilaterally misreporting her preferences because there is a continuum of workers. Further, we impose restriction (12)

We are now ready to state our main result. Our approach is to establish the existence of a stable matching that satisfies an additional property. Say that a matching M is **population-proportional** if, for each $f \in F$ and $k \in I_f$, there is some $\alpha_f^k \in [0,1]$ such that

$$M_f(\theta) = \min\{D^{\preceq f}(M)(\theta), \alpha_f^k G(\theta)\}, \quad \forall \theta \in \Theta_f^k. \tag{13}$$

In other words, the measure of workers hired by firm f from the indifference class Θ_f^k is given by the same proportion α_f^k of $G(\theta)$ for all $\theta \in \Theta_f^k$, unless the measure of worker types θ available to f is less than the proportion α_f^k of $G(\theta)$, in which case the entire available measure of that type is assigned to that firm. In short, a population-proportional matching seeks to match a firm with workers of different types in proportion to their population sizes at G whenever possible, if they belong to the same indifference class of the firm. The stability and population proportionality of a mechanism translate into the desired fairness and incentive properties, as shown by the following result.

LEMMA 2: (i) *If a matching is stable and population-proportional, then it is strongly stable.*
(ii) *If a mechanism φ implements a strongly stable matching for every measure in $\overline{\mathcal{X}}$, then the mechanism is strategy-proof for workers.*

PROOF: See Section S.8 of the Supplemental Material. *Q.E.D.*

We now present the main result of this section.

THEOREM 11: *If each firm's preference is continuous or if each firm's preference is weakly substitutable, then there exists a matching that is stable and population-proportional. Therefore, given that the domain satisfies either property, there exists a mechanism that implements a strongly stable matching and is strategy-proof.*

PROOF: See Section S.8 of the Supplemental Material. *Q.E.D.*

Recall that the workers of the same reported type receive the same ex ante assignment. By Lemma 2, strong stability and strategy-proofness will be achieved if each firm's choice were to respect population proportionality. A key step of proof is therefore to select an optimal choice $\tilde{C}_f \in C_f$ that induces population proportionality for each f. The selection $\tilde{C}_f$ is then shown to satisfy the conditions of Theorems 2 and 3, given the continuity or weak substitutability conditions. Thus, a stable matching exists in the hypothetical continuum economy in which firms have preferences represented by the choice functions $\tilde{C}_f$. The final step is to show that the stable matching of the hypothetical economy is stable in the original economy and satisfies population proportionality.

This result establishes the existence of a matching mechanism that satisfies strong stability and strategy-proofness for workers in a large economy environment.[69] In contrast to

only for types (a, P) and (a, P') that are in the support of G. For the true worker type (a, P), this is the same assumption as in the standard strategy-proofness concept for finite markets. We do not impose any condition for misreporting a measure-zero type because if φ is individually rational (which is the case for stable mechanisms), then the incentives for misreporting as a measure-zero type can be eliminated by specifying the mechanism to assign a worker reporting such a type to the null firm with probability 1.

[69]Even with a continuum of workers, no stable mechanism is strategy-proof for firms. See an example in Section S.8.2 of the Supplemental Material.

the existing literature, our result holds under general firm preferences that may involve indifferences and/or complementarities.

8.2. *Applications to Time Share/Probabilistic Matching Models*

Our model introduced in Section 8.1 has a connection with time share and probabilistic matching models. In these models, a *finite* set of workers contracts with a finite set of firms for time shares or for probabilities with which they match. Probabilistic matching is often used in allocation problems without money, such as school choice, while time share models have been proposed as a solution to labor matching markets in which part-time jobs are available (see Biró, Fleiner, and Irving (2013), for instance).

Our model in Section 8.1 can be reinterpreted as a time share model. Let Θ be the *finite* set of workers whose shares firms may contract for, *as opposed to the finite types of a continuum of workers.* The measure $G(\theta)$ represents the total endowment of time or the probability that a worker θ has available for matching. A matching M describes the time or probability $M_f(\theta)$ that a worker θ and a firm f are matched.

The partition $\mathbb{P}_f$ then describes firm f's set of indifference classes, where each class describes the set of workers that the firm considers equivalent. The function $\Lambda_f = (\Lambda_f^k)_{k\in I_f}$ describes the time shares that firm f wishes to choose from available time shares in alternative indifference classes. On the worker side, for each worker $\theta \in \Theta_P$, the first-order stochastic dominance induced by P describes the worker's preference in evaluating lotteries. With this reinterpretation, Definition 11 provides an appropriate notion of a strongly stable matching.[70] The following result is immediate:

COROLLARY 3: *The* (*reinterpreted*) *time share model admits a strongly stable—and thus stable—matching if either each firm's preference is continuous or it is weakly substitutable*.[71]

This result generalizes the existence of a strongly stable matching in the school choice problem studied by Kesten and Ünver (2014), where schools may regard multiple students as having the same priority. They showed the existence of a strongly stable probabilistic matching (which they called strong ex ante stability) under the assumption that schools have responsive preferences with ties. Our contribution is to extend the existence to general preferences that may violate responsiveness. Our result might be useful for school choice environments in which schools may need a balanced student body in terms of gender, ethnicity, income, or skill (recall footnote 3).

9. CONCLUSION

Complementarity, although prevalent in matching markets, has been known to be a source of difficulties in designing desirable mechanisms. The present paper took a step

[70]The notion of strong stability in Definition 11 requires the proportion of time spent with a firm out of the total endowment to be equalized among workers that the firm considers equivalent. This notion is sensible in the context of a time share model, particularly when $G(\theta)$ is the same across all workers, as with school choice (where each student has a unit demand). When $G(\theta)$ is different across θ's, however, one could consider an alternative notion, such as one that equalizes the absolute amount of time (not divided by $G(\theta)$) that a worker spends with a firm. Our analysis can be easily modified to prove the existence of matching that satisfies this alternative notion of strong stability.

[71]Unlike the continuum model, population proportionality does not guarantee strategy-proofness. As shown by Kesten and Ünver (2014), strategy-proofness is generally impossible to attain in time share/probabilistic models with finite numbers of workers.

toward addressing these difficulties by considering large economies. We find that complementarity need not jeopardize stability in a large market. First, as long as preferences are continuous or substitutable, a stable matching exists in a limit continuum economy. Second, with such preferences, there exists an approximately stable matching in a large finite economy. We used this framework to show that there is a stable mechanism that is strategy-proof for workers and satisfies an additional fairness property, strong stability.

The scope of our analysis can be extended in two directions. First, we can introduce "contracts," namely, allowing each firm-worker pair to match under alternative contracts, as done by Hatfield and Milgrom (2005) in the context of substitutable preferences. As in our baseline model, we focus on the measures of workers matched with alternative firms as basic units of analysis. However, unlike our main model, the measures of workers matching with a firm under *alternative contract terms* should be distinguished and thus described as a multidimensional vector. With this enrichment of the underlying space, our method can be extended to yield existence in this setup.[72] This result is provided in Section S.9 of the Supplemental Material.

Second, while we have considered the model in which a finite number of "large" firms match with a continuum of workers, we can extend our framework to study a model in which a continuum of "small" firms match with a continuum of workers, as has been studied by AH. Take their main model and for simplicity consider the pure matching case (i.e., without contracts) in which each worker demands at most one position. Like AH, assume that the set of firm types is finite. Then, one can interpret the entire mass of firms of each given type as a single "large" firm and "build" an *aggregate choice correspondence* for that fictitious large firm from optimal choices of infinitesimal firms (of the same type), say by maximizing their aggregate welfare. The aggregate choice correspondence constructed in this way is shown to satisfy the continuity condition required for the existence of a stable matching in Theorem 2. Therefore, our model can recover the existence result for a certain special case of AH. The specific result is described in Section S.10 of the Supplemental Material.

To the best of our knowledge, this paper is the first to analyze matching in a continuum economy with the level of generality presented here. As such, our paper may pose as many questions as it answers. One issue worth pursuing is the computation of a stable matching. The existence of a stable matching, as established in this paper, is clearly necessary to find a desired mechanism, but practical implementation requires an algorithm. Although our fixed-point mapping provides one such algorithm when the mapping is contractionary or monotonic (i.e., preferences are substitutable), studying the computationally efficient and generally applicable algorithms to find stable matchings would be an interesting and challenging future research topic.

APPENDIX A: PROOFS OF THEOREM 1 AND THEOREM 2

PROOF OF THEOREM 1: **("Only if" Part):** Suppose that M is a stable matching, and let $X = (X_f)_{f\in\tilde{F}}$ with $X_f = D^{\preceq f}(M)$, $\forall f \in \tilde{F}$. We prove that X is a fixed point of T. Let us first show that for each $f \in \tilde{F}$, $X_f \in \mathcal{X}$. It is clear that as each $M_{f'}(\Theta_P \cap \cdot)$ is nonnegative and

[72]Nevertheless, the generalization is more than mechanical. Since the measure of workers a firm matches with under a contract term depends on the measure of workers the same firm matches with under a different contract term, special care is needed to define the choice function and the measure of workers available to a firm under a particular contract term.

countably additive, so is $X_f(\cdot)$. It is also clear that since $(M_f)_{f\in\tilde{F}}$ is a matching, $X_f \sqsubset G$. Thus, we have $X_f \in \mathcal{X}$.

We next claim that $M_f \in C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset \sqsubset X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that $M_f \notin C_f(X_f)$, which means that there is some $M'_f \in C_f(X_f)$ such that $M'_f \neq M_f$. Note that $M_f \sqsubset X_f$ and thus $(M'_f \vee M_f) \sqsubset X_f$. Then, by revealed preference, we have $M_f \notin C_f(M'_f \vee M_f)$ and $M'_f \in C_f(M'_f \vee M_f)$, or equivalently, $M'_f \succ_f M_f$, which means that M is unstable since $M'_f \sqsubset X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X \in T(X)$. The fact that $M_f \in C_f(X_f), \forall f \in \tilde{F}$ means that $X_f - M_f \in R_f(X_f), \forall f \in \tilde{F}$. We set $Y_f = X_f - M_f$ for each $f \in \tilde{F}$ and obtain, for any $E \in \Sigma$,

$$
\begin{aligned}
&\sum_{P:P(1)=f} G(\Theta_P \cap E) + \sum_{P:P(1)\neq f} Y_{f_-^P}(\Theta_P \cap E) \\
&\quad = \sum_{P:P(1)=f} G(\Theta_P \cap E) + \sum_{P:P(1)\neq f} \big(X_{f_-^P}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E)\big) \\
&\quad = \sum_{P:P(1)=f} G(\Theta_P \cap E) + \sum_{P:P(1)\neq f} \Bigg(\sum_{f'\in\tilde{F}:f'\preceq_P f_-^P} M_{f'}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E)\Bigg) \\
&\quad = \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\preceq_P f} M_{f'}(\Theta_P \cap E) = X_f(E),
\end{aligned}
$$

where the second and fourth equalities follow from the definition of $X_{f_-^P}$ and X_f, respectively, while the third follows from the fact that f_-^P is an immediate predecessor of f and $\sum_{f'\in\tilde{F}:f'\preceq_P P(1)} M_{f'}(\Theta_P \cap E) = G(\Theta_P \cap E)$. The above equation holds for every firm $f \in \tilde{F}$, so we conclude that $X \in T(X)$, that is, X is a fixed point of T.

("If" Part): Let us first introduce some notation. Let f_+^P denote an **immediate successor** of $f \in \tilde{F}$ at $P \in \mathcal{P}$: that is, $f_+^P \prec_P f$, and for any $f' \prec_P f$, $f' \preceq_P f_+^P$. Note that for any $f, \tilde{f} \in \tilde{F}$, $f = \tilde{f}_-^P$ if and only if $\tilde{f} = f_+^P$.

Suppose now that $X \in \mathcal{X}^{n+1}$ is a fixed point of T. For each firm $f \in \tilde{F}$ and $E \in \Sigma$, define

$$
M_f(E) = X_f(E) - \sum_{P:P(n+1)\neq f} X_{f_+^P}(\Theta_P \cap E), \tag{14}
$$

where $P(n+1)$ is the least preferred firm according to P.

We first verify that for each $f \in \tilde{F}$, $M_f \in \mathcal{X}$. First, it is clear that for each $f \in \tilde{F}$, M_f is countably additive as both $X_f(\cdot)$ and $X_{f_+^P}(\Theta_P \cap \cdot)$ are countably additive. It is also clear that for each $f \in \tilde{F}$, $M_f \sqsubset X_f$. To see that $M_f(E) \geq 0, \forall E \in \Sigma$, observe from (14) that

$$
\begin{aligned}
M_f(E) &= \sum_{P:P\in\mathcal{P}} X_f(\Theta_P \cap E) - \sum_{P:P(n+1)\neq f} X_{f_+^P}(\Theta_P \cap E) \\
&\geq \sum_{P:P(n+1)\neq f} \big(X_f(\Theta_P \cap E) - X_{f_+^P}(\Theta_P \cap E)\big) \geq 0,
\end{aligned}
$$

where the inequality holds since $X \in T(X)$ means that there is some $Y_f \in R_f(X_f)$ such that $X_{f_+^P}(\Theta_P \cap \cdot) = Y_f(\Theta_P \cap \cdot)$ for each $P \in \mathcal{P}$, and thus $X_{f_+^P}(\Theta_P \cap \cdot) \sqsubset X_f(\Theta_P \cap \cdot)$.

Let us next show that for all $f \in \tilde{F}$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_f(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \tag{15}$$

which means that $X_f = D^{\preceq f}(M)$. To do so, fix any $P \in \mathcal{P}$ and consider first a firm $f = P(n+1)$ (i.e., a firm ranked lowest at P). By (14), $M_f(\Theta_P \cap E) = X_f(\Theta_P \cap E)$ and thus (15) holds for such f. Consider next any $f \neq P(n+1)$, and assume for an inductive argument that (15) holds true for f_+^P, so $X_{f_+^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f_+^P} M_{f'}(\Theta_P \cap E)$. Then, by (14), we have

$$\begin{aligned} X_f(\Theta_P \cap E) &= M_f(\Theta_P \cap E) + X_{f_+^P}(\Theta_P \cap E) = M_f(\Theta_P \cap E) + \sum_{f' \in \tilde{F}: f' \preceq_P f_+^P} M_{f'}(\Theta_P \cap E) \\ &= \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_f)_{f \in \tilde{F}}$ is a matching, let $f = P(1)$ and note that by definition of T, if $\tilde{X} \in T(X)$, then $\tilde{X}_f(\Theta_P \cap \cdot) = G(\Theta_P \cap \cdot)$. Since $X \in T(X)$, we have, for any $E \in \Sigma$,

$$G(\Theta_P \cap E) = X_f(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}} M_{f'}(\Theta_P \cap E),$$

where the second equality follows from (15). Since the above equation holds for every $P \in \mathcal{P}$, M is a matching.

We now prove that $(M_f)_{f \in \tilde{F}}$ is stable. As noted by footnote 28, the first part of Condition 1 is implied by Condition 1, which we check below. To prove the second part of Condition 1 of Definition 1, note first that $C_\emptyset(X_\emptyset) = \{X_\emptyset\}$ and thus $R_\emptyset = 0$. Fix any $P \in \mathcal{P}$ and assume $\emptyset \neq P(n+1)$, since there is nothing to prove if $\emptyset = P(n+1)$. Consider a firm f such that $f_-^P = \emptyset$. Then, X being a fixed point of T means $X_f(\Theta_P) = R_{f_-^P}(\Theta_P) = 0$, which implies by (15) that $0 = X_f(\Theta_P) = \sum_{f': f' \preceq_P f} M_{f'}(\Theta_P) = \sum_{f': f' \prec_P \emptyset} M_{f'}(\Theta_P)$, as desired.

It only remains to check Condition 1 of Definition 1. Suppose for a contradiction that it fails. Then, there exist f and M'_f such that

$$M'_f \sqsubset D^{\preceq f}(M), \qquad M'_f \in C_f\big(M'_f \vee M_f\big), \quad \text{and} \quad M_f \notin C_f\big(M'_f \vee M_f\big). \tag{16}$$

So $M'_f \sqsubset D^{\preceq f}(M) = X_f$. Since then $M_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$ and $M_f \in C_f(X_f)$, the revealed preference property implies $M_f \in C_f(M'_f \vee M_f)$, contradicting (16). We have thus proven that M is stable. *Q.E.D.*

PROOF OF THEOREM 2: We establish the compactness of $\mathcal{X}$ and the upper hemicontinuity of T in Lemma 3 and Lemma 4 below. To do so, recall that $\mathcal{X}$ is endowed with weak-$*$ topology. The notion of convergence in this topology, that is, weak convergence, can be stated as follows: A sequence of measures $(X_k)_{k \in \mathbb{N}}$ in $\mathcal{X}$ weakly converges to a measure $X \in \mathcal{X}$, written as $X_k \xrightarrow{w^*} X$, if and only if $\int_\Theta h\, dX_k \to \int_\Theta h\, dX$ for all $h \in C(\Theta)$, where $C(\Theta)$ is the space of all continuous functions defined on Θ. The next result provides some conditions that are equivalent to weak convergence.

THEOREM 12: *Let X and $(X_k)_{k\in\mathbb{N}}$ be finite measures on Σ.*[73] *The following conditions are equivalent:*[74]

(a) $X_k \xrightarrow{w^*} X$;

(b) $\int_\Theta h\,dX_k \to \int_\Theta h\,dX$ *for all* $h \in C_u(\Theta)$, *where* $C_u(\Theta)$ *is the space of all uniformly continuous functions defined on* Θ;

(c) $\liminf_k X_k(A) \geq X(A)$ *for every open set* $A \subset \Theta$, *and* $X_k(\Theta) \to X(\Theta)$;

(d) $\limsup_k X_k(A) \leq X(A)$ *for every closed set* $A \subset \Theta$, *and* $X_k(\Theta) \to X(\Theta)$;

(e) $X_k(A) \to X(A)$ *for every set* $A \in \Sigma$ *such that* $X(\partial A) = 0$ (∂A *denotes the boundary of* A).

LEMMA 3: *The space $\mathcal{X}$ is convex and compact. Also, for any $X \in \mathcal{X}$, $\mathcal{X}_X$ is compact.*

PROOF: Convexity of $\mathcal{X}$ follows trivially. To prove the compactness of $\mathcal{X}$, let $C(\Theta)^*$ denote the dual (Banach) space of $C(\Theta)$ and note that $C(\Theta)^*$ is the space of all (signed) measures on (Θ, Σ), given Θ is a compact metric space.[75] Then, by Alaoglu's theorem, the closed unit ball of $C(\Theta)^*$, denoted U^*, is weak-$*$ compact.[76] Clearly, $\mathcal{X}$ is a subspace of U^* since, for any $X \in \mathcal{X}$, $\|X\| = X(\Theta) \leq G(\Theta) = 1$. The compactness of $\mathcal{X}$ will thus follow if $\mathcal{X}$ is shown to be a closed set. To prove this, we prove that for any sequence $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ and $X \in C(\Theta)^*$ such that $X_k \xrightarrow{w^*} X$, we must have $X \in \mathcal{X}$, which will be shown if we prove that $0 \leq X(E) \leq G(E)$ for any $E \in \Sigma$. Let us first make the following observation: every finite (Borel) measure X on the metric space Θ is normal,[77] which means that for any set $E \in \Sigma$,

$$X(E) = \inf\bigl\{X(O) : E \subset O \text{ and } O \in \Sigma \text{ is open}\bigr\} \tag{17}$$

$$= \sup\bigl\{X(F) : F \subset E \text{ and } F \in \Sigma \text{ is closed}\bigr\}. \tag{18}$$

To show first that $X(E) \leq G(E)$, consider any open set $O \in \Sigma$ such that $E \subset O$. Then, since $X_k \in \mathcal{X}$ for every k, we must have $X_k(O) \leq G(O)$ for every k, which, combined with (c) of Theorem 12 above, implies that $X(O) \leq \liminf_k X_k(O) \leq G(O)$. Given (17), this means that $X(E) \leq G(E)$.

To show next that $X(E) \geq 0$, consider any closed set $F \in \Sigma$ such that $F \subset E$. Since $X_k \in \mathcal{X}$ for every k, we must have $X_k(F) \geq 0$, which, combined with (d) of Theorem 12 above, implies that $X(F) \geq \limsup_k X_k(F) \geq 0$. Given (18), this means $X(E) \geq 0$.

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[73]We note that this result can be established without having to assume that X is nonnegative, as long as all X_k's are nonnegative.

[74]This theorem is a modified version of "Portmanteau theorem" that is modified to deal with any finite (i.e., not necessarily probability) measures. See Theorem 2.8.1 of Ash and Doléans-Dade (2009) for this result, for instance.

[75]More precisely, $C(\Theta)^*$ is isometrically isomorphic to the space of all signed measures on (Θ, Σ) according to the Riesz representation theorem (see Royden and Fitzpatrick (2010), for instance).

[76]The closed unit ball is defined as $U^* := \{X \in C^*(\Theta) : \|X\| \leq 1\}$, where $\|X\|$ is the dual norm, that is,

$$\|X\| = \sup\left\{\left|\int_\Theta h\,dX\right| : h \in C(\Theta) \text{ and } \max_{\theta\in\Theta}\bigl|h(\theta)\bigr| \leq 1\right\}.$$

If X is a nonnegative measure, then the supremum is achieved by taking $h \equiv 1$, and thus $\|X\| = X(\Theta)$. It is well known (see Royden and Fitzpatrick (2010), for instance) that if $C(\Theta)^*$ is infinite dimensional, then U^* is not compact under the norm topology (i.e., the topology induced by the dual norm). On the other hand, U^* is compact under the weak-$*$ topology, which follows from Alaoglu's theorem (see Royden and Fitzpatrick (2010), for instance).

[77]See Theorem 12.5 of Aliprantis and Border (2006).

The proof for the compactness of $\mathcal{X}_X$ is analogous and hence omitted. *Q.E.D.*

LEMMA 4: *The map T is a correspondence from $\mathcal{X}^{n+1}$ to itself. Also, it is nonempty- and convex-valued, and upper hemicontinuous.*

PROOF: To show that T maps from $\mathcal{X}^{n+1}$ to itself, observe that for any $X \in \mathcal{X}^{n+1}$ and $\tilde{X} \in T_f(X)$, there is $Y_f \in R_f(X_f)$ for each $f \in \tilde{F}$ such that, for all $E \in \Sigma$,

$$\tilde{X}(E) = \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap E) \leq \sum_{P\in\mathcal{P}} X_{f_-^P}(\Theta_P \cap E) \leq \sum_{P\in\mathcal{P}} G(\Theta_P \cap E) = G(E),$$

which means that $\tilde{X} \in \mathcal{X}$, as desired. (Here and from now on, we adopt the convention that for any measure X, $X_{f_-^P} = G$ when $P(1) = f$.)

As noted earlier, the correspondence T is nonempty-valued. To prove that T is convex-valued, it suffices to show that, for each $f \in \tilde{F}$, R_f is convex-valued. Consider any $X \in \mathcal{X}$ and $Y', Y'' \in R_f(X)$. There are some $X', X'' \in C_f(X)$ satisfying $Y' = X - X'$ and $Y'' = X - X''$. Then, the convexity of $C_f(X)$ implies that, for any $\lambda \in [0, 1]$, $\lambda X' + (1-\lambda)X'' \in C_f(X)$ so $\lambda Y' + (1-\lambda)Y'' = X - (\lambda X' + (1-\lambda)X'') \in R_f(X)$.

To establish the upper hemicontinuity of T, we first establish the following claim:

CLAIM 1: *For any sequence $(X_k)_{k\in\mathbb{N}} \subset \mathcal{X}$ that weakly converges to $X \in \mathcal{X}$, a sequence $(X_k(\Theta_P \cap \cdot))_{k\in\mathbb{N}}$ also weakly converges to $X(\Theta_P \cap \cdot)$ for all $P \in \mathcal{P}$.*

PROOF: Let X^P and X_k^P denote $X(\Theta_P \cap \cdot)$ and $X_k(\Theta_P \cap \cdot)$, respectively. Note first that for any $X \in \mathcal{X}$, we have $X^P \in \mathcal{X}$ for all $P \in \mathcal{P}$. Due to Theorem 12, it suffices to show that X^P and $(X_k^P)_{k\in\mathbb{N}}$ satisfy Condition (c) of Theorem 12. To do so, consider any open set $O \subset \Theta$. Then, letting Θ_P° denote the interior of Θ_P,

$$\begin{aligned}\liminf_k X_k^P(O) &= \liminf_k X_k\big(\Theta_P^\circ \cap O\big) + X_k(\partial\Theta_P \cap O)\\ &= \liminf_k X_k\big(\Theta_P^\circ \cap O\big) \geq X\big(\Theta_P^\circ \cap O\big) = X^P(O),\end{aligned}$$

where the second equality follows from the fact that $X_k(\partial\Theta_P \cap O) \leq X_k(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$, the lone inequality from $X_k \xrightarrow{w^*} X$, (c) of Theorem 12, and the fact that $\partial\Theta_P^\circ \cap O$ is an open set, and the last equality from repeating the first two equalities with X instead of X_k. Also, we have

$$X_k^P(\Theta) = X_k(\Theta_P) \to X(\Theta_P) = X^P(\Theta),$$

where the convergence is due to $X_k \xrightarrow{w^*} X$, (e) of Theorem 12, and the fact that $X(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$. Thus, the two requirements in Condition (c) of Theorem 12 are satisfied, so $X_k^P \xrightarrow{w^*} X^P$, as desired. *Q.E.D.*

It is also straightforward to observe that if C_f is upper hemicontinuous, then R_f is also upper hemicontinuous.

We now prove the upper hemicontinuity of T by considering any sequences $(X_k)_{k\in\mathbb{N}}$ and $(\tilde{X}_k)_{k\in\mathbb{N}}$ in $\mathcal{X}^{n+1}$ weakly converging to some X and $\tilde{X}$ in $\mathcal{X}^{n+1}$, respectively, such that $\tilde{X}_k \in T(X_k)$ for each k. To show that $\tilde{X} \in T(X)$, let $X_{k,f}$ and $\tilde{X}_{k,f}$ denote the components of X_k and $\tilde{X}_k$, respectively, that correspond to $f \in \tilde{F}$. Then, we can find $Y_{k,f} \in R_f(X_{k,f})$ for each k and f such that

$$\tilde{X}_{k,f}(\cdot) = \sum_{P \in \mathcal{P}} Y_{k,f_-^P}(\Theta_P \cap \cdot). \tag{19}$$

Consider a converging subsequence $(Y_{k(m),f})_{m\in\mathbb{N}}$ of $(Y_{k,f})_{k\in\mathbb{N}}$, and its limit Y_f for each firm $f \in \tilde{F}$. (Note that such a subsequence must exist since $(Y_{k,f})$ lies in the compact set $\mathcal{X}$.) By Claim 1, $Y_{k(m),f}(\Theta_P \cap \cdot) \xrightarrow{w^*} Y_f(\Theta_P \cap \cdot)$ for all $P \in \mathcal{P}$. Given this and (19), we have $\tilde{X}_{k(m),f}(\cdot) \xrightarrow{w^*} \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot)$, which implies $\tilde{X}_f(\cdot) = \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot)$ since $(\tilde{X}_{k,f})$, and thus $(\tilde{X}_{k(m),f})$, converges to $\tilde{X}_f$. Meanwhile, we have $Y_f \in R_f(X_f)$ since R_f is upper hemicontinuous, since $(X_{k(m),f})$ and $(Y_{k(m),f})$ converge to X_f and Y_f, respectively, and since $Y_{k(m),f} \in R_f(X_{k(m),f})$ for all m. We thus conclude that $\tilde{X} \in T(X)$, as desired. *Q.E.D.*

Lemmas 3 and 4 show that T is nonempty- and convex-valued, and upper hemicontinuous while it is a mapping from the convex, compact space $\mathcal{X}^{n+1}$ into itself, which implies that T is also closed-valued. Thus, we can invoke Kakutani–Fan–Glicksberg's fixed-point theorem to conclude that the mapping T has a nonempty set of fixed points. *Q.E.D.*

APPENDIX B: PROOFS FOR SECTION 6

PROOF OF THEOREM 3: Recall from Lemma S1 that the partially ordered set $(\mathcal{X}, \sqsubset)$, and thus the partially ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$, is a complete lattice, where $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ if $X_f \sqsubset X'_f$ for all $f \in \tilde{F}$. If each C_f is closed-valued, so are each R_f and T, as one can easily verify. Also, if each R_f is weak-set monotonic, so is T in the ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$. Note also that $\mathcal{X}^{n+1}$ is a compact set due to Lemma 3. Thus, if all firms have weakly substitutable preferences with closed-valued choice correspondences, then T has a fixed point according to Corollary 3.7 of Li (2014), which implies existence of a stable matching due to Theorem 1. *Q.E.D.*

PROOF OF THEOREM 4: *Proof of Part (i):* Note first that by substitutability, each R_f is weak-set monotonic while $R_f(X)$ is a complete sublattice for any $X \in \mathcal{X}$, and that these properties are inherited by T. Given this, the proof of Theorem 1 in Zhou (1994) shows that the set of fixed points of T, denoted $\mathcal{X}^*$, contains the largest and smallest elements, $\overline{X} = \sup_{\sqsubset_{\tilde{F}}} \mathcal{X}^*$ and $\underline{X} = \inf_{\sqsubset_{\tilde{F}}} \mathcal{X}^*$.[78] Let $\overline{M}$ and $\underline{M}$ be the stable matchings associated with $\overline{X}$ and $\underline{X}$, respectively, given by Theorem 1. We only establish that $\overline{M}$ is firm-optimal and worker-pessimal, since the result for $\underline{M}$ can be established analogously. Recall from our characterization theorem that for any stable matching M, there is some $X \in \mathcal{X}^*$ such that $X_f = D^{\preceq f}(M)$ and $M_f \in C_f(X_f)$ for all $f \in \tilde{F}$. We thus have $M_f \sqsubset X_f \sqsubset \overline{X}_f$, which implies that $\overline{M}_f \in C_f(M_f \vee \overline{M}_f)$ by revealed preference since $\overline{M}_f \in C_f(\overline{X}_f)$ and $(M_f \vee \overline{M}_f) \sqsubset \overline{X}_f$. Thus, $\overline{M}_f \succeq_f M_f$ for each $f \in F$, as desired. To show that $\overline{M} \preceq_\Theta M, \forall M \in \mathcal{M}^*$, fix any

[78] Zhou (1994)'s theorem requires the strong-set monotonicity, but some inspection of its proof reveals that the weak-set monotonicity is sufficient for existence of largest and smallest fixed points.

$M \in \mathcal{M}^*$ and consider $X \in \mathcal{X}^*$ such that $X_f = D^{\preceq f}(M)$ for all $f \in \tilde{F}$. Then, for all $P \in \mathcal{P}$ and $f \in \tilde{F} \backslash \{P(n+1)\}$,

$$X_{f_+^P}(\Theta_P \cap E) = D^{\preceq f_+^P}(M)(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} M_{f'}(\Theta_P \cap E), \quad \forall E \in \Sigma,$$

where f_+^P is an immediate successor of $f \in \tilde{F}$ at $P \in \mathcal{P}$, as defined earlier. Similarly, for $\overline{X}$, we have $\overline{X}_{f_+^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} \overline{M}_{f'}(\Theta_P \cap E)$. Given this and the fact that $X \sqsubset_{\tilde{F}} \overline{X}$,

$$\begin{aligned}\sum_{f' \in \tilde{F}: f' \succeq_P f} \overline{M}_{f'}(\Theta_P \cap E) &= G(\Theta_P \cap E) - \overline{X}_{f_+^P}(\Theta_P \cap E) \\ &\leq G(\Theta_P \cap E) - X_{f_+^P}(\Theta_P \cap E) \qquad (20) \\ &= \sum_{f' \in \tilde{F}: f' \succeq_P f} M_{f'}(\Theta_P \cap E)\end{aligned}$$

for all $E \in \Sigma$, and for all $P \in \mathcal{P}$ and $f \in \tilde{F} \backslash \{P(n+1)\}$. Note that (20) also holds for $f = P(n+1)$ since both LHS and RHS are then equal to $G(\Theta_P \cap E)$.

Proof of Part (ii): Note that for any $X \in \mathcal{X}$, each $R_f(X)$, and thus $T(X)$, is a complete sublattice. Then, $\overline{T}$ must be monotonic since, for any $X \sqsubset X'$, we have $\overline{T}(X) \in T(X)$ and $\overline{T}(X') \in T(X')$, which implies by upper weak-set monotonicity that there exists $Y \in T(X')$ such that $\overline{T}(X) \sqsubset Y$, and then $\overline{T}(X) \sqsubset \overline{T}(X')$ by the definition of $\overline{T}(X')$. Now let $X^0 = (X_f^0)_{f \in \tilde{F}}$ with $X_f^0 = G, \forall f \in \tilde{F}$. Define recursively $X^n = \overline{T}(X^{n-1})$ for each $n \geq 1$. The sequence $(X^n)_{n \in \mathbb{N}}$ is decreasing since $X^1 \sqsubset X^0$ and $X^2 = \overline{T}(X^1) \sqsubset \overline{T}(X^0) = X^1$ and so on, which implies that it has a limit point, denoted X^*. Because each C_f is order-continuous, we have $\overline{R}_f(X_f^n) = X_f^n - \underline{C}_f(X_f^n) \xrightarrow{w^*} X_f^* - \underline{C}_f(X_f^*) = \overline{R}_f(X_f^*)$, which implies that $X^{n+1} = \overline{T}(X^n) \xrightarrow{w^*} \overline{T}(X^*)$. Since $X^{n+1} \xrightarrow{w^*} X^*$, we must have $\overline{T}(X^*) = X^*$. To show that $X^* = \overline{X}$, consider any $X \in \mathcal{X}^*$. Then, $X \sqsubset X^0$ and thus $X \sqsubset \overline{T}(X) \sqsubset \overline{T}(X^0) = X^1$. Repeating this way, we obtain $X \sqsubset X^n, \forall n$, which implies that $X \sqsubset X^*$ and thus $X^* = \overline{X}$. By the proof of Part (i), a stable matching associated with $\overline{X}$ is firm-optimal. The proof for worker-optimal stable matching is analogous and thus omitted. *Q.E.D.*

PROOF OF THEOREM 5: Let M be any stable matching. Then, by Theorem 1, there exists $X \in \mathcal{X}^*$ such that $M_f \in C_f(X_f)$ for each $f \in F$. Since $X \sqsubset_{\tilde{F}} \overline{X}$, LoAD implies

$$\overline{M}_f(\Theta) \geq \inf C_f(\overline{X}_f)(\Theta) \geq \sup C_f(X_f)(\Theta) \geq M_f(\Theta), \quad \forall f \in F. \qquad (21)$$

Next since $\overline{M}$ is worker-pessimal, (20) holds for any $f \in \tilde{F}$. Let $w_P := \emptyset_-^P$ be the immediate predecessor of $\emptyset$ (i.e., the worst individually rational firm) for types in Θ_P. Then, setting $f = w_P$ in (20), we obtain

$$\begin{aligned}\sum_{f' \in F} \overline{M}_{f'}(\Theta_P \cap E) &= \sum_{f' \in \tilde{F}: f' \succeq_P w_P} \overline{M}_{f'}(\Theta_P \cap E) \\ &\leq \sum_{f' \in \tilde{F}: f' \succeq_P w_P} M_{f'}(\Theta_P \cap E) = \sum_{f' \in F} M_{f'}(\Theta_P \cap E), \quad \forall E \in \Sigma,\end{aligned}$$

or equivalently,

$$\sum_{f'\in F}\overline{M}_{f'}(E)\le\sum_{f'\in F}M_{f'}(E),\quad \forall E\in\Sigma. \tag{22}$$

Since this inequality must hold with $E=\Theta$, combining it with (21) implies that $M_f(\Theta)=\overline{M}_f(\Theta)$ for all $f\in F$, as desired.

Further, we must have $\sum_{f\in F}\overline{M}_f=\sum_{f\in F}M_f$. Otherwise, by (22), we must have $\sum_{f'\in F}\overline{M}_{f'}(E)<\sum_{f'\in F}M_{f'}(E)$ for some $E\in\Sigma$. Also, by (22), $\sum_{f'\in F}\overline{M}_{f'}(E^c)\le\sum_{f'\in F}M_{f'}(E^c)$. Combining these two inequalities, we obtain $\sum_{f'\in F}\overline{M}_{f'}(\Theta)<\sum_{f'\in F}M_{f'}(\Theta)$, which contradicts (21). Last, $\sum_{f\in F}\overline{M}_f=\sum_{f\in F}M_f$ means $\overline{M}_\emptyset=M_\emptyset$. *Q.E.D.*

PROOF OF THEOREM 6: Suppose otherwise. Then there exists a stable matching M that differs from the worker-optimal stable matching $\underline{M}$. Let X and $\underline{X}$ be respectively fixed points of T such that $M_f=C_f(X_f)$, $\underline{M}_f=C_f(\underline{X}_f)$, and $\underline{X}_f\sqsubset X_f$, for each $f\in F$.

First of all, since $\underline{X}_f\sqsubset X_f$ for each $f\in F$, we have $(\underline{M}_f\vee M_f)\sqsubset X_f$. Revealed preference then implies that, for each $f\in F$,

$$M_f=C_f(\underline{M}_f\vee M_f)$$

or $M\succeq_F\underline{M}$. Moreover, since $M\neq\underline{M}$, the set $\bar{F}:=\{f\in F|M_f\succ_f\underline{M}_f\}$ is nonempty. But then by the rich preferences, there exists $f^*\in F$ such that

$$\underline{M}_{f^*}\neq C_{f^*}\big((\underline{M}_{f^*}+M_{\bar{F}}^{f^*})\wedge G\big).$$

For each $f\in F\setminus\bar{F}$, $M_f=\underline{M}_f$, by the definition of $\bar{F}$, and Theorem 5 guarantees that $M_\emptyset=\underline{M}_\emptyset$. Consequently, we have, for each $E\in\Sigma$, that

$$M_{\bar{F}}^{f^*}(E)=\sum_{P\in\mathcal{P}}\sum_{f':f^*\succ_P f',f'\notin\bar{F}}M_{f'}(\Theta_P\cap E)=\sum_{P\in\mathcal{P}}\sum_{f':f^*\succ_P f',f'\notin\bar{F}}\underline{M}_{f'}(\Theta_P\cap E)=\underline{M}_{\bar{F}}^{f^*}(E).$$

It then follows that $(\underline{M}_{f^*}+M_{\bar{F}}^{f^*})\wedge G=(\underline{M}_{f^*}+\underline{M}_{\bar{F}}^{f^*})\wedge G=\underline{M}_{f^*}+\underline{M}_{\bar{F}}^{f^*}$ (since $\underline{M}$ is a matching), so

$$\underline{M}_{f^*}\neq C_{f^*}\big(\underline{M}_{f^*}+\underline{M}_{\bar{F}}^{f^*}\big). \tag{23}$$

Letting $\hat{M}_{f^*}:=C_{f^*}(\underline{M}_{f^*}+\underline{M}_{\bar{F}}^{f^*})$, we have $\hat{M}_{f^*}\sqsubset(\underline{M}_{f^*}\vee\hat{M}_{f^*})\sqsubset(\underline{M}_{f^*}+\underline{M}_{\bar{F}}^{f^*})$. Revealed preference then implies that

$$\hat{M}_{f^*}=C_{f^*}(\underline{M}_{f^*}\vee\hat{M}_{f^*}).$$

Then, by (23), we have $\hat{M}_{f^*}\succ_{f^*}\underline{M}_{f^*}$. Further, $\hat{M}_{f^*}\sqsubset(\underline{M}_{f^*}+\underline{M}_{\bar{F}}^{f^*})\sqsubset D^{\preceq f^*}(\underline{M})$. We therefore have a contradiction to the stability of $\underline{M}$. *Q.E.D.*

APPENDIX C: PROOFS FOR SECTION 7

PROOF OF THEOREM 7: Let Γ be the limit continuum economy to which the sequence $(\Gamma^q)_{q\in\mathbb{N}}$ converges. For any population G, fix a sequence $(G^q)_{q\in\mathbb{N}}$ of finite-economy populations such that $G^q \xrightarrow{w^*} G$. Let $\Theta^q = \{\theta_1^q, \theta_2^q, \dots, \theta_{\bar{q}}^q\} \subset \Theta$ be the support for G^q.[79] For each firm $f \in \tilde{F}$, define Θ_f to be the set of types that find firm f acceptable, that is, $\Theta_f := \bigcup_{P \in \mathcal{P}: f \succ_P \emptyset} \Theta_P$ (let $\Theta_\emptyset = \Theta$ by convention). Let $\overline{\Theta}_f$ denote the closure of Θ_f with respect to the topology on Θ. We first prove a few preliminary results, whose proofs are provided in Section S.7.1 of the Supplemental Material.

LEMMA 5: *For any $r > 0$, there is a finite number of open balls, $B_1, \dots, B_L$, in Θ that have radius smaller than r with a boundary of zero measure (i.e. $G(\partial B_\ell) = 0, \forall \ell$) and cover $\overline{\Theta}_f$ for each $f \in F$.*

LEMMA 6: *Consider any $X, Y \in \mathcal{X}$ such that $X(\Theta \backslash \Theta_f) = 0$ for some $f \in \tilde{F}$ and $X \sqsubset Y$, and consider any sequence $(Y^q)_{q \in \mathbb{N}}$ such that $Y^q \in \mathcal{X}^q$ and $Y^q \xrightarrow{w^*} Y$.*[80] *Then, there exists a sequence $(X^q)_{q \in \mathbb{N}}$ such that $X^q \in \mathcal{X}^q$, $X^q \xrightarrow{w^*} X$, $X^q \sqsubset Y^q$, and $X^q(\Theta \backslash \Theta_f) = 0$ for all q.*

LEMMA 7: *For any two sequences $(X^q)_{q \in \mathbb{N}}$ and $(Y^q)_{q \in \mathbb{N}}$ such that $X^q, Y^q \in \mathcal{X}^q$, $X^q \sqsubset Y^q, \forall q$, $X^q \xrightarrow{w^*} X$, and $Y^q \xrightarrow{w^*} Y$, we have $X \sqsubset Y$.*

Using these lemmas, we establish the following two lemmas:

LEMMA 8: *For any stable matching M in Γ and $\epsilon > 0$, there is $Q \in \mathbb{N}$ such that, for any $q > Q$, one can construct a matching $M^q = (M_f^q)_{f \in \tilde{F}}$ that is feasible and individually rational in Γ^q, and satisfies*

$$u_f(M_f) < u_f(M_f^q) + \frac{\epsilon}{2}, \quad \forall f \in F. \tag{24}$$

PROOF: In any finite economy Γ^q, let us construct a matching $\tilde{M}^q = (\tilde{M}_f^q)_{f \in \tilde{F}}$ as follows: order the firms in F by $f_1, \dots, f_n$, and

1. define $\tilde{M}_{f_1}^q$ as X^q in Lemma 6 with $X = M_{f_1}$, $Y = G$, and $Y^q = G^q$;[81]
2. define $\tilde{M}_{f_2}^q$ as X^q in Lemma 6 with $X = M_{f_2}$, $Y = G - M_{f_1}$, and $Y^q = G^q - \tilde{M}_{f_1}^q$ (this is possible since $G^q - \tilde{M}_{f_1}^q \xrightarrow{w^*} G - M_{f_1}$);
3. in general, for each $f_k \in \tilde{F}$, define inductively $\tilde{M}_{f_k}^q$ as X^q in Lemma 6 with $X = M_{f_k}$, $Y = G - \sum_{k' < k} M_{f_{k'}}$, and $Y^q = G^q - \sum_{k' < k} \tilde{M}_{f_{k'}}^q$;

and define $\tilde{M}_\emptyset^q = G^q - \sum_{f \in F} \tilde{M}_f^q$.

By Lemma 6, $\tilde{M}^q$ is feasible in Γ^q and individually rational for workers while $\tilde{M}^q \xrightarrow{w^*} M$. To ensure the individual rationality for firms, we construct another matching $M^q = (M_f^q)_{f \in \tilde{F}}$ as follows: for each $f \in F$, select any $M_f^q \in C_f^q(\tilde{M}_f^q)$, and then set $M_\emptyset^q = G^q - \sum_{f \in F} M_f^q$. By revealed preference, we have $M_f^q \in C_f^q(M_f^q)$ and thus M^q is individually rational for firms. Also, the individual rationality of M^q for workers follows immediately from the individual rationality of $\tilde{M}^q$ and the fact that $M_f^q \sqsubset \tilde{M}_f^q$ for all $f \in F$. By the

[79]Note that we allow for the possibility that there are multiple workers of the same type even in finite economies, so $\bar{q}$ may be strictly smaller than q.

[80]Note that if $f = \emptyset$, then $\Theta \backslash \Theta_f = \emptyset$. Thus, the restriction that $X(\Theta \backslash \Theta_f) = 0$ becomes vacuous.

[81]Note that $M_f(\Theta \backslash \Theta_f) = 0$ for all $f \in F$ since M is individually rational, so Lemma 6 can be applied.

continuity of u_f's and the fact $\tilde{M}_f^q \xrightarrow{w^*} M_f$, we can find sufficiently large $Q \in \mathbb{N}$ such that, for all $q > Q$,

$$u_f(M_f) < u_f(\tilde{M}_f^q) + \frac{\epsilon}{2} \leq u_f(M_f^q) + \frac{\epsilon}{2}, \quad \forall f \in F,$$

where the second inequality holds since $M_f^q \in C_f^q(\tilde{M}_f^q)$. *Q.E.D.*

LEMMA 9: *The matching M^q constructed in Lemma 8 is ϵ-stable in Γ^q for all $q > Q$, where Q is identified in Lemma 8.*

PROOF: Let $D^{\preceq f}(M^q)$ be the subpopulation of workers in economy Γ^q who weakly prefer f to their match in M^q.[82] Since $M^q \xrightarrow{w^*} M$, we have $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$.[83] Choose any $\tilde{M}_f^q \in C_f(D^{\preceq f}(M^q))$. In words, $\tilde{M}_f^q$ is the most profitable block of M^q for f in the continuum economy, that is, the optimal deviation in a situation where the current matching is M^q, but the firm can deviate to any subpopulation, not just a discrete distribution. Then, we must have

$$u_f(\tilde{M}_f^q) < u_f(M_f) + \frac{\epsilon}{2}, \tag{25}$$

for any sufficiently large q. Otherwise, we could find some subsequence $(\hat{M}_f^q)_{q\in\mathbb{N}}$ of sequence $(\tilde{M}_f^q)_{q\in\mathbb{N}}$ for which

$$u_f(\hat{M}_f^q) \geq u_f(M_f) + \frac{\epsilon}{2}, \quad \forall q. \tag{26}$$

We can assume that $(\hat{M}_f^q)_{q\in\mathbb{N}}$ is converging to some $\hat{M}_f$ (by choosing a further subsequence if necessary). Then, the above-mentioned property that $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$ and upper hemicontinuity of C_f imply $\hat{M}_f \in C_f(D^{\preceq f}(M))$ and thus $u_f(\hat{M}_f) = u_f(M_f)$ since $M_f \in C_f(D^{\preceq f}(M))$ (due to stability of M), which contradicts (26).

Now let M_f' be the most profitable block of M^q for f in economy Γ^q. Then, M_f' is the optimal deviation facing the same population G^q and matching M^q as when computing $\tilde{M}_f^q$ but with an additional restriction that the deviation is feasible in Γ^q (multiples of $1/q$), so $u_f(M_f') \leq u_f(\tilde{M}_f^q)$. This and inequality (25) imply

$$u_f(M_f') < u_f(M_f) + \frac{\epsilon}{2}. \tag{27}$$

Combining inequalities (24) and (27), we get $u_f(M_f') < u_f(M_f^q) + \epsilon$, completing the proof. *Q.E.D.*

Theorem 7 then follows from the existence of stable matching M in Γ and Lemmas 8 and 9. *Q.E.D.*

[82]To be precise, $D^{\preceq f}(M^q)$ is given as in (4) with G and X being replaced by G^q and M^q, respectively.

[83]This convergence can be shown using an argument similar to that which we have used to establish the continuity of Ψ in the proof of Lemma 4.

PROOF OF THEOREM 8: The proof that M is a matching in Γ is straightforward and thus omitted. We first show that M is individually rational. First of all, since M^q is individually rational for workers, we have $M_f^q(\Theta_P) = 0$ for all $f \in F$ and $P \in \mathcal{P}$ such that $\emptyset \succ_P f$, which implies that $M_f(\Theta_P) = 0$ since $M_f^q \xrightarrow{w^*} M_f$ and $M_f(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$. Thus, M is also individually rational for workers. To show that M is individually rational for firms, suppose for a contradiction that there are some $f \in F$ and $\hat{M}_f \in \mathcal{X}$ such that $\hat{M}_f \sqsubset M_f$ and $u_f(\hat{M}_f) - u_f(M_f) = 3\epsilon$ for some $\epsilon > 0$. We then prove the following claim:

CLAIM 2: *For all sufficiently large q, there exists a subpopulation $\hat{M}_f^q$ in Γ^q such that $\hat{M}_f^q \sqsubset D^{\preceq f}(M^q)$ and $u_f(\hat{M}_f^q) > u_f(\hat{M}_f) - \epsilon$.*

PROOF: We use Lemma 6 with $Y = D^{\preceq f}(M)$, $Y^q = D^{\preceq f}(M^q)$, and $X = \hat{M}_f$. By the continuity of $D^{\preceq f}(\cdot)$ and the assumption that $M^q \xrightarrow{w^*} M$, we have $Y^q \xrightarrow{w^*} Y$. Also, we have $X = \hat{M}_f \sqsubset M_f \sqsubset D^{\preceq f}(M) = Y$. Lemma 6 then implies that there exists a sequence $(\hat{M}_f^q)_{q\in\mathbb{N}}$ such that $\hat{M}_f^q \in \mathcal{X}^q$, $\hat{M}_f^q \xrightarrow{w^*} X = \hat{M}_f$, and $\hat{M}_f^q \sqsubset Y^q = D^{\preceq f}(M^q)$. Then, by the continuity of u_f, we have $u_f(\hat{M}_f^q) > u_f(\hat{M}_f) - \epsilon$ for all sufficiently large q. *Q.E.D.*

Since $M^q \xrightarrow{w^*} M$ and u_f is continuous, we have that, for all sufficiently large q,

$$u_f(M_f^q) < u_f(M_f) + \epsilon = u_f(\hat{M}_f) - 2\epsilon < u_f(\hat{M}_f^q) - \epsilon, \tag{28}$$

where the second inequality follows from Claim 2. This contradicts ϵ-stability of M^q in Γ^q.

To prove that there is no blocking coalition, suppose for a contradiction that there exist a firm $f \in F$ and subpopulation $\hat{M}_f$ such that $\hat{M}_f \sqsubset D^{\preceq f}(M)$ and $u_f(\hat{M}_f) - u_f(M_f) = 3\epsilon$ for some $\epsilon > 0$. By Claim 2, for all sufficiently large q, there exists a subpopulation $\hat{M}_f^q$ in Γ^q such that $\hat{M}_f^q \sqsubset D^{\preceq f}(M^q)$ and $u_f(\hat{M}_f^q) > u_f(\hat{M}_f) - \epsilon$. Then, the same inequality as in (28) establishes the desired contradiction. *Q.E.D.*

Let us here state a variant of Theorem 8 for later use, whose proof is essentially the same as that of Theorem 8:

THEOREM 1: Let $(M^{q_k})_{k\in\mathbb{N}}$ be a sequence of matchings converging to M such that for every $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that, for all $k > K$, M^{q_k} is ϵ-stable in Γ^{q_k}. Then, M is stable in Γ.

PROOF OF THEOREM 9: First let us state a mathematical fact:

LEMMA 10—Heine–Cantor Theorem: *Let $h : A \to B$ be a continuous function between two metric spaces A and B, and suppose A is compact. Then, h is uniformly continuous.*

Since the space of all subpopulations of G is metrizable by the Lévy–Prokhorov metric, and it is compact, the Heine–Cantor theorem applies to our setting.

We also need the following result:

LEMMA 11: *For every $\epsilon > 0$, there exist $\delta \in (0, \epsilon)$ and $Q' \in \mathbb{N}$ such that, for every $q > Q'$ and every matching M^q that is δ-stable in Γ^q, there exists a stable matching M in Γ such that $d(M^q, M) < \epsilon$, where $d(\cdot, \cdot)$ is the product Lévy–Prokhorov metric.*[84]

PROOF: Suppose for contradiction that the conclusion of the statement does not hold. Then there exists $\epsilon > 0$ with the following property: for every $\delta \in (0, \epsilon)$ and $Q' \in \mathbb{N}$, there exist $q > Q'$ and M^q that is δ-stable in Γ^q such that $d(M^q, M) \geq \epsilon$ for every M that is stable in Γ. This implies there exists a decreasing sequence $(\delta^k)_k$ which converges to 0 and $(M^{q^k})_k$ such that M^{q^k} is δ^k-stable in Γ^{q^k}, $d(M^{q^k}, M) \geq \epsilon$ for every stable matching M in Γ, and $\lim_k q^k = \infty$. Without loss of generality, assume M^{q^k} converges to some matching $\hat{M}$ (because the sequence lies in a sequentially compact space). Then $d(\hat{M}, M) \geq \epsilon$ for every stable matching $M \in \Gamma$, so $\hat{M}$ is not stable in Γ. This is a contradiction to Theorem 1. *Q.E.D.*

Proof of Part (i): Given an arbitrary $\epsilon > 0$, let $\eta > 0$ be a constant such that, for any two matchings M and M', $d(M, M') < \eta$ implies $|u_f(M_f) - u_f(M'_f)| < \epsilon/2$ for every $f \in F$. (Recall that u_f is continuous. Therefore, it is uniformly continuous by the Heine–Cantor theorem.) Without loss, one can assume $\eta < \epsilon$.

For $\eta > 0$ defined in the last paragraph, choose $\delta \in (0, \eta)$ and Q' as described in the statement of Lemma 11. (Note that $\delta < \epsilon$ since $\delta < \eta < \epsilon$.) More precisely, δ and Q' have the property that for every $q > Q'$ and every matching $\hat{M}^q$ that is δ-stable in Γ^q, there exists a stable matching M in Γ such that $d(\hat{M}^q, M) < \eta$. Given this δ, by Lemma 8 and Lemma 9, there is $Q > Q'$ such that, for all $q > Q$, there exists a matching M^q in Γ^q which is δ-stable in Γ^q and satisfies

$$u_f(M_f^q) > u_f(\overline{M}_f) - \frac{\delta}{2} > u_f(\overline{M}_f) - \frac{\epsilon}{2}. \tag{29}$$

CLAIM 3: *$u_f(\overline{M}_f) > u_f(\hat{M}_f^q) - \epsilon/2$ for any δ-stable matching $\hat{M}^q$ in Γ^q.*

PROOF: By the argument in the last paragraph, there exists a stable matching M in Γ with $d(\hat{M}^q, M) < \eta$. So, by construction of η (and uniform continuity of u_f), we obtain $u_f(M_f) > u_f(\hat{M}_f^q) - \epsilon/2$. Meanwhile, by firm optimality of $\bar{M}$, we have $u_f(M_f) \leq u_f(\overline{M}_f)$. Combining these inequalities, we obtain the desired inequality. *Q.E.D.*

Then, the desired conclusion holds for any $q > Q$ since, by (29) and Claim 3, we have $u_f(M_f^q) > u_f(\overline{M}_f) - \epsilon/2 > u_f(\hat{M}_f^q) - \epsilon$.

Proof of Part (ii): Note first that each mapping $D^{\succeq f}(\cdot)$ is continuous, and hence uniformly continuous (see footnote 83). Thus, given an arbitrary $\epsilon > 0$, one can choose $\eta \in (0, \epsilon)$ such that, for any $M, M' \in \mathcal{X}^{n+1}$, $d(M, M') < \eta$ implies $d(D^{\succeq f}(M), D^{\succeq f}(M')) < \frac{\epsilon}{2}$ for all $f \in \tilde{F}$. By Lemma 11, for the chosen η, one can find $\delta \in (0, \eta)$ and $Q' \in \mathbb{N}$ such that,

[84]The Lévy–Prokhorov metric on space $\mathcal{X}$ is defined as follows: for any $X, Y \in \mathcal{X}$,

$$d(X, Y) := \inf\{\varepsilon > 0 | X(E) \leq Y(E^\varepsilon) + \varepsilon \text{ and } Y(E) \leq X(E^\varepsilon) + \varepsilon \text{ for all } E \in \Sigma\},$$

where $E^\epsilon := \{\theta \in \Theta | \exists \theta' \in E \text{ such that } d^\Theta(\theta, \theta') < \epsilon\}$ with d^Θ being a metric for the space Θ. Here, we abuse notation since d is used to denote both the Lévy–Prokhorov metric and its product metric. Note that the choice of product metric is inconsequential since it is defined on a finite-dimensional space.

for every $q > Q'$ and every δ-stable matching $\hat{M}^q$ in Γ^q, there is a stable matching $\tilde{M}^q$ in Γ such that $d(\hat{M}^q, \tilde{M}^q) < \eta$. By definition of η, we must have $d(D^{\succeq f}(\tilde{M}^q), D^{\succeq f}(\hat{M}^q)) < \frac{\epsilon}{2}$.

Next, given that $C_f(\underline{M}_f) = \{\underline{M}_f\}$ for each $f \in F$, Lemma S5 of the Supplemental Material implies that there is a sequence $(M^q)_{q\in\mathbb{N}}$ such that $M^q \xrightarrow{w^*} \underline{M}$, where M^q is a feasible and individually rational matching in Γ^q. Choose now $\epsilon_\delta > 0$ such that, for any subpopulations $M, M' \in \mathcal{X}$, $d(M, M') < \epsilon_\delta$ implies $|u_f(M) - u_f(M')| < \delta$. By Lemma S6 of the Supplemental Material, one can find $Q'' \in \mathbb{N}$ such that for all $q > Q''$, M^q is ϵ_δ-distance stable: that is, for any $M' \in \mathcal{X}^q$ such that $M' \sqsubset D^{\preceq f}(M^q)$ and $u_f(M') > u_f(M^q_f)$, we have $d(M', M^q_f) < \epsilon_\delta$. This implies by the definition of ϵ_δ that $u_f(M^q_f) + \delta > u_f(M')$. In other words, M^q is δ-stable for all $q > Q''$, as required by Condition 1 of Definition 10. To satisfy Condition 2, using the fact that M^q converges to $\underline{M}$, we can choose $Q > \max\{Q', Q''\}$ such that, for all $q > Q$, we have $d(D^{\succeq f}(M^q), D^{\succeq f}(\underline{M})) < \frac{\epsilon}{2}$ for all $f \in \tilde{F}$, which implies

$$D^{\succeq f}(\underline{M})(E) \leq D^{\succeq f}(M^q)(E^{\frac{\epsilon}{2}}) + \frac{\epsilon}{2}, \forall E \in \Sigma, \quad \forall f \in \tilde{F}, \tag{30}$$

by the fact that d is the Lévy–Prokhorov metric (refer to footnote 84 for the definition of d and E^ε). Then, for any $q > Q$ and for any $f \in \tilde{F}$ and $E \in \Sigma$,

$$\begin{aligned}\left(D^{\succeq f}(\hat{M}^q)(E) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} &\leq D^{\succeq f}(\tilde{M}^q)(E^{\frac{\epsilon}{2}}) - \frac{\epsilon}{2} \\ &\leq D^{\succeq f}(\underline{M})(E^{\frac{\epsilon}{2}}) - \frac{\epsilon}{2} \leq D^{\succeq f}(M^q)((E^{\frac{\epsilon}{2}})^{\frac{\epsilon}{2}}) \leq D^{\succeq f}(M^q)(E^\epsilon),\end{aligned}$$

where the first inequality follows since $d(D^{\succeq f}(\tilde{M}^q), D^{\succeq f}(\hat{M}^q)) < \frac{\epsilon}{2}$, the second inequality from the worker-optimality of $\underline{M}$ and stability of $\tilde{M}^q$ in Γ, the third inequality from (30), and the last inequality from the fact that $(E^{\frac{\epsilon}{2}})^{\frac{\epsilon}{2}} \subset E^\epsilon$ (which can be easily verified). *Q.E.D.*

PROOF OF THEOREM 10: Suppose not for contradiction. Then, there must be a sequence $(\delta_k, q_k)_{k\in\mathbb{N}}$ with $\delta_k \searrow 0$ and $q_k \nearrow \infty$ such that $\hat{M}^{q_k}$ is δ_k-stable and $d(M, \hat{M}^{q_k}) \geq \epsilon$ for all k. Then, one can find a subsequence $(q_{k_m})_{m\in\mathbb{N}}$ such that $\hat{M}^{q_{k_m}}$ converges to some $\hat{M}$ (since the sequence lies in a sequentially compact space), which must be stable in Γ due to Theorem 1. Since $d(M, \hat{M}^{q_{k_m}}) \geq \epsilon$ for all m, we must have $d(M, \hat{M}) \geq \epsilon$, which contradicts the uniqueness of stable matching in Γ. *Q.E.D.*

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